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Metastability and Convergence to Equilibrium for the Random Field Curie–Weiss Model

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We study a dynamics for the magnetization of the random field Curie–Weiss model. A metastable behavior is exhibited and asymptotic estimates on the speed of convergence to equilibrium are given. The results are given almost surely and in law with respect to the realizations of the random magnetic fields.

KEY WORDS: Metastability; random magnetic field; random spin systems; Glauber dynamics; Curie–Weiss model.

1. INTRODUCTION

The study of disordered statistical mechanics is one of the most important subject of the last decade in mathematical physics. A lot of progresses have been done on the equilibrium properties of these systems. Even if the problem of the spin glass phase is still a challenging one, some disordered systems are well understood. Good examples are random fields models. Rigorous results concerning the controversial lower critical dimension d_l (above which there is symmetry breaking) of the random field Ising model were given by J. Bricmont and A. Kupiainen[16] in 1988, they proved that long ranged order exists for the random field Ising model in dimension greater or equal 3, which implies that $d_l \leqslant 2$. It was later proved by M. Aizenman and J. Wehr[2] that such a long range order does not exist in dimension 2, this implies that $d_l \geqslant 2$, hence $d_l = 2$. This proved that the Imry–Ma scaling argument[36] was in fact correct. Previous convincing arguments in this direction had been given by D. Fisher, J. Fröhlich, and T. Spencer[27] and by J. Chalker.[20] J. Imbrie[35] had proved that at zero

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temperature the ground state is ordered for $d=3$, a result that suggests that $d_l \leqslant 2$. One dimensional models with short range interactions have been studied by A. Beretti,[8] P. Bleher, J. Ruiz, and V. Zagrebnov[9] and with long range interactions by M. Aizenman and J. Wehr[2] where absence of long range order is proved for one dimensional models with interactions between spins at sites i and j decaying like $|i-j|^{3/2+\varepsilon}$.

Mean field theory is a nice way to make a simple model that presents a symmetry breaking phenomenon, see R. Ellis.[26] The Curie–Weiss model is a simple model to explain the phenomena of ferromagnetism, that is the appearence of a spontaneous magnetization at low temperature. Even if mean fields models in general display unphysical properties, they have always been a good starting point to test new ideas. The mean field versions of random fields models were first considered by T. Schneider and E. Pytte[61] with a Gaussian distribution for the magnetic fields. The discrete distribution case was first considered by A. Aharony.[1] The complete study of the phase diagram of this model is given by S. Salinas and W. Wreszinski,[60] the fluctuations of the order parameter are studied by J. Amaro de Matos and J. Perez,[3] and the Gibbs states by J. Amaro de Matos, A. Patrick, and V. Zagrebnov.[4] In a very recent paper, C. Külske[37] studies the notion of metastates, introduced by C. Newman and D. Stein.[52–54] The Random field Curie–Weiss model has the advantage to be sufficiently rich and to be almost completely solvable. There are other mean field models for disordered systems. The Sherrington and Kirkpatrick model was invented to explain the spin glass phase transition, it is still a challenging problem to solve it, see refs. 48 and 63. Another famous disordered mean field model is the Hopfield model, a lot of progresses have been done the last decade by A. Bovier and V. Gayrard and A. Bovier, V. Gayrard, and P. Picco.[10–15] There is a multibody version of the Sherrington and Kirkpatrick model called the Random Energy Model that was introduced by B. Derrida,[23, 24] and a lot of results have been obtained by E. Olivieri and P. Picco[56] and A. Galves, S. Martìnez, and P. Picco.[30] All these results concern static properties. The Dynamics for the Curie–Weiss model was studied by R. Griffiths, C.-H. Weng, and J. Langer[32] and in a rigorous way in the fundamental paper of M. Cassandro, A. Galves, E. Olivieri, and M.-E. Varés.[18] Dynamics for mean field disordered systems was studied by G. Ben Arous and A. Guionnet for a soft spins version of the Sherrington and Kirkpatrick model,[6, 7] and for a discrete spin version by M. Grunwald,[33] however this is for fixed time scale. The McKean–Vlasov limit of a dynamics for Curie–Weiss random field Ising model was obtained by P. Dai Pra and F. Den Hollander.[21] There, also, the time scale is fixed. In a recent article L. R. Fontes, M. Isopi, Y. Koyashawa, and P. Picco[28] study the convergence to equilibrium of the Random Energy Model, by giving asymptotic estimates of the spectral gap, they work on a time scale which is of the order of an exponential in the volume.

Another important subject of modern statistical mechanics is the study of metastable behavior of stochastic spin systems. See the book of T. Liggett[40] for historical facts and basic background on stochastic spin systems. The notion of metastable behavior is a very old one and comes for experiments on the liquid-vapor transition in the middle of the last century. The arguments of Van der Waals against the Maxwell modification of the Van der Waals equation of state were based on the existence of metastable states that are observed experimentaly as supersaturated vapor and supercooled liquid. The metastable behavior is a dynamical behavior, where a system remains for a very long time in an apparent equilibrium, which is called the metastable state, then very quickly relaxes to the true equilibrium state. On the correct time-scale, the process therefore behaves like a pure jump process with two states.

The rigorous formulation of metastable states was given by O. Penrose and J. L. Lebowitz.[58] A new point of view was introduced by M. Cassandro, A. Galves, E. Olivieri, and M. E. Varés[18] under the name of "Pathwise approach to metastability." As a basic example, they establish the metastability of the Curie–Weiss model. Even if it is the unphysical non convexity of the canonical free energy which is responsible of the metastable behavior of this model, the pathwise approach proved to be an efficient concept to describe dynamical properties of more realistic models. Indeed metastability was later proved for random perturbations of dynamical systems of Freidlin and Wentzel type[29] by A. Galves, E. Olivieri, and M. E. Varés[31] in finite dimension and by M. Cassandro, E. Olivieri, and P. Picco,[19] F. Martinelli, E. Olivieri, and E. Scoppola[43] and S. Brassesco[8] in the infinite dimensional case. The pathwise approach to metastability was extended to spin systems as the two dimensional stochastic Ising model by R. Schonmann and J. Neves,[55] for the Swendsen–Wang dynamic by F. Martinelli, E. Olivieri, and E. Scoppola,[41] the generalization to other 2 dimensional lattice systems was done by R. Kotecky and E. Olivieri,[38, 39] E. Cirillo and E. Olivieri[25] and F. Nardi and E. Olivieri.[51] Metastability of the three dimensional Ising model has been studied by G. Ben Arous and R. Cerf.[5] All those results are for a fixed large volume, a fixed magnetic field h and vanishing temperature (β very large). The case of large volume, β fixed and $h \downarrow 0$ was studied by R. Schonmann.[62]

The usual technics used to prove metastability are based on either explicit computations or on large deviation estimates. Explicit computations of moments of hitting times for a birth and death process related to the dynamics of the deterministic Curie–Weiss model are possible in ref. 18 only because we are in a one-dimensional situation. As we shall see, to describe the random field Curie–Weiss model, it is necessary to introduce two order parameters. Therefore the technics used by ref. 18 are not directly applicable. Large deviation technics can often yield sharp estimates on hitting times. For instance a mixture of large deviation and partial differential equations technics is used to prove the exponentiality of some exit times by M. Williams[64] and M. Day.[22] However we chose a different, and in some sense more direct, approach based on spectral estimates. Remember that we want to show that our process has the following behavior: it quickly reaches the metastable state, remains there for a long time and then jumps to the invariant state. The idea of the proof is that it is sufficient to show that, on the correct time scale, only two states remain. Since the markovian nature of the process cannot be lost, we automatically get that the jump from one state to the other one occurs at an exponential time, i.e., the unpredictability property of the escape from metastability. Using a spectral decomposition of the generator of the process, one can express its law as a linear combination of eigenvectors. Proving that only two states remain amounts to proving that only two terms remain in this sum. This is achieved via estimates of the eigenvalues. At this point we obtain a rather abstract statement: the process is indeed metastable but the metastable state is given in terms of some eigenvector and the time scale is given as an eigenvalue. With a little more work we identify the metastable state. It is also shown that, on the exponential scale, the escape time is given by the activation energy. Let us stress the fact that eigenvalues are natural objects to consider: indeed the spectral gaps we compute are relaxation times for the metastable and the stable states. Dirichlet eigenvalues are related to hitting times, in our case, the escape time from metastability. The technic we adopt to estimate eigenvalues was introduced by R. Holley, S. Kusuoka, and D. Stroock[34] to study simulated annealing. It was extended to get not only the spectral gap but also other eigenvalues and estimates on eigenvectors by P. Mathieu[45] and in L. Miclo[49] where discrete and continuous state spaces are considered but rather strong regularity are imposed. P. Mathieu was able to obtain long time asymptotics for extremely irregular potentials as Wiener medium including metastability statements (See refs. 45, 46, and 47). The same approach was further developed by L. Miclo[50] for a discrete state space. Our aim has been to compute as little as possible and to reduce the problem of metastability to a static problem as quickly as possible. In particular we want to avoid computations on the fluctuations of the hamiltonian due to the randomness of the external field. As opposed to the large deviation point of view, we directly study the process on the metastable scale, in particular, we do not describe the behavior of the process on a finite (fixed) time interval. The estimates we need are very robust because eigenvalues are not sensitive to small variations of the free energy. This point is important: at the microscopic level, the hamiltonian is very irregular because of the disorder of the external field, but as the number of particules increases, it becomes smooth as a consequence of the law of large numbers. We also take advantage of the Markov property to deduce the exponentiality of the escape time directly from rough asymptotics. The fact that the processes we consider are all reversible plays an important role.

For this model, the hamiltonian is random but, as a consequence of the law of large numbers, the free energy is deterministic. As far as static properties are concerned, the metastable state is deterministic. Although the support of the stable state is also a deterministic two points space, the repartition of the weights of the stable state on its support is random, i.e., depends on the fluctuations of the external field. Similarly the metastable scale is random although its order on the exponential scale is deterministic. It is one of the interesting features of this model to be able to discuss precisely the impact of the disorder of the medium on the dynamical properties of the process.

The paper is organized as follows. In Chapter 2 we define the model and the dynamics we consider. We state the main results. In Chapter 3 we give asymptotic for various spectral quantities. In Chapter 4 we prove some results for the static. In Chapter 5 we prove the theorems In the appendix we make a lenghtly computation that link the Dirichlet form of the dynamics on the spins to the ones induced on the empirical order parameter. The reader might find more convenient to skip Sections 3 and 4 and go directly to the proof of the theorems in Section 5, assuming the results of the previous sections.

2. THE MODEL AND MAIN RESULTS

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space on which is defined a family of i.i.d.r.v. $(h_i)_{i \in \mathbb{N}}$ with a Bernoulli distribution, that is $\mathbb{P}[h_1 = +1] = \mathbb{P}[h_1 = -1] = 1/2$. Let $\mathscr{S}_N = \{-1, +1\}^N$ and, given $\sigma \in \mathscr{S}_N$ and $(h_i)_{i=1}^N = (h_i(\omega))_{i=1}^N$, let us define the random Hamiltonian:

$$H_N(\sigma) \equiv H_N(\sigma, \omega) = -\frac{N}{2}\left(\frac{1}{N} \sum_{i=1}^N \sigma_i\right)^2 - \theta \sum_{i=1}^N h_i(\omega)\, \sigma_i \tag{2.1}$$

Since most of the quantities that will appear in this paper are random variables on Ω, we will never make explicit the ω dependence of this Hamiltonian, nor the one of the magnetic fields.

Let us denote by μ_N the Gibbs measure on $\mathscr{S}_N$:

$$\mu_N(\sigma) = \exp[-\beta H_N(\sigma)]/Z_N \tag{2.2}$$

here Z_N, the partition function in a volume N, is the normalization factor to turn μ_N into a probability measure on $\mathscr{S}_N$.

Let us define the dynamics we consider. Given $N \in N$, let L_N be the operator acting on real valued functions ϕ on $\mathscr{S}_N$:

$$L_N\phi(\sigma) = \frac{1}{N} \sum_{i=1}^{N} (\phi(\sigma^i) - \phi(\sigma))\, e^{-\beta/2[H_N(\sigma^i) - H_N(\sigma)]} \tag{2.3}$$

here σ^i is the configuration obtained from σ by making a spin flip at the site i, that is $\sigma^i_j = \sigma_j$ for $j \neq i$ and $\sigma^i_i = -\sigma_i$. For future use, we will denote also $T^i\sigma \equiv \sigma^i$. The continuous time Markov process, $\sigma_N(t)$ defined on D $(\mathbb{R}^+, \mathscr{S}_N)$, the space of càdlàg function on $\mathbb{R}^+$ with value in $\mathscr{S}_N$, is then defined by the transition functions $P^N_t(\sigma \to \sigma') \equiv P^N_t(\sigma, \sigma')$ determined by the forward equation (the Fokker–Planck equation)

$$\frac{\partial}{\partial t} P^N_t(\sigma, \sigma') = [L^*_N P^N_t(\sigma, \cdot)](\sigma') \tag{2.4}$$

with $P_0(\sigma, \sigma') = \mathbb{1}_\sigma(\sigma')$, here $\mathbb{1}_\sigma(\sigma') = 0$ if $\sigma' \neq \sigma$, $\mathbb{1}_\sigma(\sigma) = 1$. In (2.4), L^*_N is the adjoint of L_N, in fact (2.4) means, $\forall t > 0$

$$\frac{\partial}{\partial t} [P^N_t \phi](\sigma) = [P^N_t L_N \phi](\sigma) \tag{2.5}$$

for real valued ϕ on $\mathscr{S}_N$ and

$$[P^N_t \phi](\sigma) = \sum_{\sigma' \in \mathscr{S}_N} P^N_t(\sigma, \sigma')\, \phi(\sigma') \tag{2.6}$$

It is easy to check that this dynamics is reversible with respect to the Gibbs measure, that is we have

$$-\sum_{\sigma \in \mathscr{S}_N} \psi(\sigma)(L_N\phi)(\sigma)\, \mu_N(\sigma) = -\sum_{\sigma \in \mathscr{S}_N} (L_N\psi)(\sigma)\, \phi(\sigma)\, \mu_N(\sigma) \tag{2.7}$$

i.e., the operator L_N is symmetric on the space $L^2(\mathscr{S}_N, \mu_N)$.

Then we have $P^N_t(\sigma, \sigma') = (e^{tL_N} \mathbb{1}_{\sigma'})(\sigma)$.

One of the important features of mean field models is that the Hamiltonian depends only on few parameters, in the Curie–Weiss model this parameter is the empirical magnetization. For the Random field model there are in fact two parameters that we introduce now;

$$
\begin{aligned}
m_N^+(\sigma) &= \frac{1}{N} \sum_{i=1}^{N} \frac{(1+h_i)}{2} \sigma_i \\
m_N^-(\sigma) &= \frac{1}{N} \sum_{i=1}^{N} \frac{(1-h_i)}{2} \sigma_i
\end{aligned} \tag{2.8}
$$

that is $m_N^+(\sigma)$ (resp. $m_N^-(\sigma)$) is the empirical magnetization on the sites where the magnetic field is positive (resp. negative). Note that we do not normalize these quantities with the number of sites where the magnetic field has a given sign but with the total number of sites. This choice has some advantages The point is that the Hamiltonian (2.1) can be written in term of these variables as

$$
\begin{aligned}
H_N(\sigma) &= H_N(m_N^+(\sigma), m_N^-(\sigma)) \\
&= -N[\tfrac{1}{2}(m_N^+(\sigma) + m_N^-(\sigma))^2 + \theta(m_N^+(\sigma) - m_N^-(\sigma))]
\end{aligned} \tag{2.9}
$$

and is constant on spin configurations σ that give the same value to the pair $m_N^+(\sigma), m_N^-(\sigma))$.

Therefore, calling $\mathcal{M}_N$, the random map from $\mathcal{S}_N$ into $[-1, +1]^2$: $\sigma \to (m_N^+(\sigma), m_N^-(\sigma))$, it is natural to consider the measure induced by this map. Since this map is discrete valued, we need more definitions to characterize its range as a discrete random subset of $[-1, +1]^2$. Let $N_+ = N_+(\omega, N) = \{i : h_i = +1\}$ and $N_- = N_-(\omega, N) = \{i : h_i = -1\}$, and note that

$$
\begin{aligned}
|N_+| &= \sum_{i=1}^{N} \frac{(1+h_i)}{2} \\
|N_-| &= \sum_{i=1}^{N} \frac{(1-h_i)}{2}
\end{aligned} \tag{2.10}
$$

we call $\rho_N^+ = \rho_N^+(\omega) = |N_+|/N$ and $\rho_N^- = \rho_N^-(\omega) = |N_-|/N$. Note that $\rho_N^+ + \rho_N^- = 1$. Moreover we have

$$
\begin{aligned}
m_N^+(T^i\sigma) &= m_N^+(\sigma^i) = m_N^+(\sigma) - (1+h_i)\frac{\sigma_i}{N} \\
m_N^-(T^i\sigma) &= m_N^-(\sigma^i) = m_N^-(\sigma) - (1+h_i)\frac{\sigma_i}{N}
\end{aligned} \tag{2.11}
$$

that is if $i \in N_+$

$$m_N^+(T^i\sigma) = m_N^+(\sigma) - \frac{2\sigma_i}{N}$$
$$m_N^-(T^i\sigma) = m_N^-(\sigma) \tag{2.12}$$

and if $i \in N_-$

$$m_N^+(T^i\sigma) = m_N^+(\sigma)$$
$$m_N^-(T^i\sigma) = m_N^-(\sigma) - \frac{2\sigma_i}{N} \tag{2.13}$$

Now it is easy to check that if we call, with a little abuse of notation, $\mathscr{M}_N \equiv \mathscr{M}_N(\omega) = \mathscr{M}_N \mathscr{S}_N$ the range of the the random map $\mathscr{M}_N$, we have

$$\mathscr{M}_N = \left\{ -\rho_N^+, -\rho_N^+ + \frac{2}{N}, \ldots, \rho_N^+ \right\} \times \left\{ -\rho_N^-, -\rho_N^- + \frac{2}{N}, \ldots, \rho_N^- \right\} \tag{2.14}$$

For a given pair $(m^+, m^-) \in \mathscr{M}_N$ we have

$$\sum_{\sigma \in \mathscr{S}_N} \mathbb{1}_{\{m_N^+(\sigma) = m^+,\, m_N^-(\sigma) = m^-\}} = \binom{N^+}{\left(1 + \frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1 + \frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}} \tag{2.15}$$

Therefore, if we denote by $\mathscr{G}_N$ the measure induced by the map $\mathscr{M}_N$, we have the following explicit formula for the density of $\mathscr{G}_N$ with respect to the uniform measure on $\mathscr{M}_N$:

$$\mathscr{G}_N(m^+, m^-) = \frac{e^{-\beta N \mathscr{F}_N(m^+, m^-)}}{Z_N} \tag{2.16}$$

here

$$\mathscr{F}_N(m^+, m^-) = -\frac{1}{2}(m^+ + m^-)^2 - \theta(m^+ - m^-)$$
$$- \frac{1}{\beta N} \log \binom{N^+}{\left(1 + \frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1 + \frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}} \tag{2.17}$$

Note that all the effects of the randomness coming from the magnetic field are present in the fact that the set $\mathscr{M}_N$ is a random subset of

$[-1, +1]^2$ and the only random parameter is ρ_N^+. This factor converges $\mathbb{P}$-almost surely to 1/2, therefore, the limit $N \uparrow \infty$, of $\mathscr{F}_N$ is the function $\mathscr{F}$, defined on $[-1, +1]^2$.

$$\mathscr{F}(m) = -\frac{1}{2}(m^+ + m^-)^2 - \theta(m^+ - m^-) + \frac{1}{2\beta}(I(2m^+) + I(2m^-)) \quad (2.18)$$

here for $x \in [-1, +1]$,

$$I(x) = \frac{1+x}{2}\log\frac{1+x}{2} + \frac{1-x}{2}\log\frac{1-x}{2} \quad (2.19)$$

and for $|x| \geqslant 1$, $I(x) = 0$, is the entropy of Bernoulli random variables.

The critical points of $\mathscr{F}$ satisfy

$$\begin{aligned} m^+ &= \tfrac{1}{2}\tanh(\beta(m^+ + m^-) + \beta\theta) \\ m^- &= \tfrac{1}{2}\tanh(\beta(m^+ + m^-) - \beta\theta) \end{aligned} \quad (2.20)$$

Setting $\bar{m} = m^+ + m^-$ we get that $\bar{m}$ has to satisfy

$$\bar{m} = \tfrac{1}{2}[\tanh(\beta\bar{m} + \beta\theta) + \tanh(\beta\bar{m} - \beta\theta)] \quad (2.21)$$

It is not difficult to see that if θ is large enough, then the only solution is $m = 0$ and there exists a curve $\theta = \theta(\beta)$ such that if the parameters $\beta \geqslant 0$, $\theta \geqslant 0$ are above this curve and θ is small enough, then there are three solutions to the equation (2.21), one is $m = 0$ and the two others are $m = m_* \equiv m_*(\beta, \theta)$ and $m = -m_*$. We will restrict ourself to this parameter region. In this region of parameters, the three critical points are

$$\begin{aligned} m_0 &= (\tfrac{1}{2}\tanh(\beta\theta), -\tfrac{1}{2}\tanh(\beta\theta)) \\ m_1 &= (\tfrac{1}{2}\tanh(\beta m_* + \beta\theta), \tfrac{1}{2}\tanh(\beta m_* - \beta\theta)) \\ m_2 &= (\tfrac{1}{2}\tanh(-\beta m_* + \beta\theta), -\tfrac{1}{2}\tanh(\beta m_* + \beta\theta)) \end{aligned} \quad (2.22)$$

Note that m_0 is a saddle point and belongs to the line $m^+ + m^- = 0$, m_1 is one absolute minima and belongs to the half plane $m^+ + m^- > 0$, m_2 being the other minima. The basin of attraction of m_1 is the triangle $T_1 = [-1/2, +1/2]^2 \cap \{m^+ + m^- > 0\}$. We define the cost in free energy to leave the basin of attraction of m_1

$$\Delta\mathscr{F} \equiv \mathscr{F}(m_0) - \mathscr{F}(m_1) = \mathscr{F}(m_0) - \mathscr{F}(m_2) \quad (2.23)$$

This quantity is fundamental for the study of metastable properties of the model. It is called in the physic literature the *activation energy*.

We want to consider the dynamics on $\mathcal{M}_N$, induced by the dynamics on $\mathcal{S}_N$ and by the previous map $\mathcal{M}_N$. Since we were unable to find a reference even for the simpler Curie–Weiss model, we make a part of the computations here, another one will be done in the appendix.

We will check that $\mathcal{M}_N(\sigma_N(t))$ is a Markov process with generator $\mathcal{L}_N$ in $L^2(\mathcal{M}_N, \mathcal{G}_N)$ given by

$$(\mathcal{L}_N\phi)(m) = \sum_{\substack{\tilde{m}\in\mathcal{M}_N\\ \tilde{m}\sim m}} [\phi(\tilde{m}) - \phi(m)]\, \mathcal{N}_N(\tilde{m}, m)\, e^{-\beta/2[H_N(\tilde{m}) - H_N(m)]} \tag{2.24}$$

and Dirichlet form with respect to $\mathcal{G}_N$

$$\begin{aligned}\mathcal{E}_N(\Psi, \Psi) &= -\mathcal{G}_N(\Psi[\mathcal{L}_N\Psi])\\ &= \frac{1}{2Z_N} \sum_{\substack{\tilde{m}, m\in\mathcal{M}_N\\ \tilde{m}\sim m}} (\Psi(\tilde{m}) - \Psi(m))^2\\ &\quad\times(\tilde{\mathcal{N}}_N(\tilde{m}, m))^{1/2}\, e^{-\beta N/2\,[\mathcal{F}_N(\tilde{m}) + \mathcal{F}_N(m)]}\end{aligned} \tag{2.25}$$

with domain the set of real valued functions on $\mathcal{M}_N$. In (2.24) and (2.25), $\tilde{m}\sim m$ means that $\tilde{m}$ and m communicate, that is, if $\tilde{m} = (\tilde{m}^+, \tilde{m}^-)\in\mathcal{M}_N$ and $m = (m^+, m^-)\in\mathcal{M}_N$, $\tilde{m}\sim m$ if we have, see (2.12) and (2.13),

$$\tilde{m}_N = \left(m_N^+(\sigma) - \frac{(1+\varepsilon_1)\,\varepsilon_2}{N},\, m_N^-(\sigma) - \frac{(1-\varepsilon_1)\,\varepsilon_2}{N}\right) \tag{2.26}$$

for a unique pair $(\varepsilon_1, \varepsilon_2)\in\{-1, +1\}^2$. The term $\mathcal{N}_N(\tilde{m}, m)$ is just

$$\mathcal{N}_N(\tilde{m}, m) = \frac{(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} \tag{2.27}$$

and $\tilde{\mathcal{N}}_N(\tilde{m}, m)$ is just

$$\tilde{\mathcal{N}}_N(\tilde{m}, m) \equiv \frac{(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2}\, \frac{(-\varepsilon_2 m^{\varepsilon_1} + 2/N + \rho_N^{\varepsilon_1})}{2} \tag{2.28}$$

Note that $\mathcal{N}_N$ is bounded from above by 1 and is bounded from below by $2/N$ if $m\in\mathcal{M}_N$ and $\tilde{m}_N\in\mathcal{M}_N$, in fact $\mathcal{N}_N$ is zero if we attend to make a jump outside $\mathcal{M}_N$. The same is true for $\tilde{\mathcal{N}}_N$. As we see later they are completely irrelevant on the time scales we are considering. In fact they even cancel an irrelevant subdominant term that is diverging on the boundary of $\mathcal{M}_N$, this subdominant term comes from the Stirling formula that will be used for giving the asymptotic behavior of (2.15). The presence of these terms is what make the difference between the dynamics on the empirical order parameter m_N induced by the dynamics on the spins and another dynamics directly defined on $\mathcal{M}_N$ which is also reversible with respect to the canonical Gibbs measure $\mathcal{G}_N$. For this later dynamics, the factor $\mathcal{N}_N(\tilde{m}, m)$ is not present in the generator nor $\tilde{\mathcal{N}}_N$ in the Dirichlet form. Note that ref. 18 chose this later dynamics in their study of the usual Curie–Weiss model, while ref. 32 chose the very same one as we did.

We want to check that the image by the map $\mathcal{M}_N$ of the spin dynamics is still a markovian dynamics with generator defined in (2.24) and Dirichlet form defined in (2.25). To do that, let $\mathcal{I}_N$ be the set of real valued functions on $\mathcal{S}_N$ that depend only on the magnetization (m_N^+, m_N^-). We just have to check that the semi-group e^{tL_N} leaves $\mathcal{I}_N$ invariant. It is enough to check that the infinitesimal generator L_N leaves $\mathcal{I}_N$ invariant. Given $\tilde{\sigma}$ and σ we say that $\tilde{\sigma}$ communicates with σ if $\tilde{\sigma} = \sigma^i$ for some $i \in \{1,..., N\}$. This will be denoted $\tilde{\sigma} \sim \sigma$.

Let $\phi \in \mathcal{I}_N$, and $\sigma \in \mathcal{S}_N$, then by (2.3) we have, if m_N is such that $\mathcal{M}_N(\sigma) = m_N$

$$
\begin{aligned}
(L_N\phi)(\sigma)
&= \frac{1}{N} \sum_{\tilde{\sigma} : \tilde{\sigma} \sim \sigma} [\phi(\tilde{\sigma}) - \phi(\sigma)]\, e^{-\beta/2\{H_N(\tilde{\sigma}) - H_N(\sigma)\}} \\
&= \frac{1}{N} \sum_{\tilde{\sigma} : \tilde{\sigma} \sim \sigma} [\phi(\mathcal{M}_N(\tilde{\sigma})) - \phi(\mathcal{M}_N(\sigma))]\, e^{-\beta/2\{H_N(\mathcal{M}_N(\tilde{\sigma})) - H_N(\mathcal{M}_N(\sigma))\}} \\
&= \sum_{\tilde{m} : \tilde{m} \in \mathcal{M}_N} \frac{1}{N} \sum_{\tilde{\sigma} : \tilde{\sigma} \sim \sigma} \mathbb{1}_{\mathcal{M}_N(\tilde{\sigma})}(\tilde{m}) [\phi(\tilde{m}) - \phi(\mathcal{M}_N(\sigma))] \\
&\quad \times e^{-\beta/2\{H_N(\tilde{m}) - H_N(\mathcal{M}_N(\sigma))\}} \\
&= \sum_{\tilde{m} : \tilde{m} \sim \mathcal{M}_N(\sigma)} [\phi(\tilde{m}) - \phi(\mathcal{M}_N(\sigma))]\, e^{-\beta/2\{H_N(\tilde{m}) - H_N(\mathcal{M}_N(\sigma))\}} \\
&\quad \times \left[\frac{1}{N} \sum_{\tilde{\sigma} : \tilde{\sigma} \sim \sigma} \mathbb{1}_{\mathcal{M}_N(\tilde{\sigma})}(\tilde{m}) \right]
\end{aligned}
\tag{2.29}
$$

Therefore, if for any $m_N \in \mathcal{M}_N$, and $\sigma \in \mathcal{S}_N$ such that $m_N = \mathcal{M}_N(\sigma)$ and any $\tilde{m} \sim m_N$ the quantity

$$
\frac{1}{N} \sum_{\tilde{\sigma} : \tilde{\sigma} \sim \sigma} \mathbb{1}_{\mathcal{M}_N(\tilde{\sigma})}(\tilde{m}) \tag{2.30}
$$

does not depend on the particular σ that realises $m_N = \mathscr{M}_N(\sigma)$, (or what is the same is constant on the set of realizations σ such that $\mathscr{M}_N(\sigma) = m_N$), we get that for any $\phi \in \mathscr{I}_N$, $L_N\phi$ depends only on m_N that is L_N leaves $\mathscr{I}_N$ invariant. To check it, note that if $\tilde{m} \sim \mathscr{M}_N(\sigma)$ then there exists an unique pair $(\varepsilon_1, \varepsilon_2) \in \{-1, +1\}^2$ such that

$$\tilde{m}_N = \left(m_N^+(\sigma) - \frac{(1+\varepsilon_1)\,\varepsilon_2}{N},\ m_N^-(\sigma) - \frac{(1-\varepsilon_1)\,\varepsilon_2}{N} \right) \tag{2.31}$$

and also it is easy to check by inspection that

$$\frac{1}{N} \sum_{\tilde{\sigma}\,:\,\tilde{\sigma} \sim \sigma} \mathbb{1}_{\mathscr{M}_N(\tilde{\sigma})}(\tilde{m}) = \frac{(\varepsilon_2 m_N^{\varepsilon_1}(\sigma) + \rho_N^{\varepsilon_1})}{2} = \mathscr{N}_N(\tilde{m}, \mathscr{M}_N(\sigma)) \tag{2.32}$$

which depends only on (m_N^+, m_N^-) and $\tilde{m}$ and not on the particular σ that realises $m_N = \mathscr{M}_N(\sigma)$, which is what we wanted to check.

That is, we can define the operator $\mathscr{L}_N$ on the set of real valued functions on $\mathscr{M}_N$ by: $[\mathscr{L}_N\phi](m_N) = [L_N\phi](m_N(\sigma))$ if σ is such that $m_N = m_N(\sigma)$. It is immediate that this operator is symmetric. The fact that the associated Dirichlet form is given by (2.25) is a little more lenghtly to check and it is done in the appendix.

Therefore we get that the continuous time Markov Process $m_N(t)$, image by the map $\mathscr{M}_N$ of the Markov process $\sigma(t)$ is the Markov process, with trajectories in $D\,[\mathbb{R}^+, \mathscr{M}_N]$, the set of càdlàg $\mathscr{M}_N$ valued functions, and transition functions given by

$$P_t^N[m_N, \tilde{m}_N] = (\exp\{t\mathscr{L}_N\}\ 1_{\tilde{m}_N})(m_N) \tag{2.33}$$

We used the same symbols for the dynamics on the spins and for the dynamics on the parameters m_N since the former one will be not used anymore. Note, in particular, that from the previous computations we get

$$\begin{aligned} P_t^N[m, \tilde{m}] &\equiv P_t^N[m \to \tilde{m}] \\ &= \sum_{\tilde{\sigma} \in \mathscr{S}_N} P_t^N(\sigma, \tilde{\sigma})\ \mathbb{1}_{\{m_N^+(\tilde{\sigma}) = \tilde{m}^+,\ m_N^-(\tilde{\sigma}) = \tilde{m}^-\}} \end{aligned} \tag{2.34}$$

here, σ is any configuration in $\mathscr{S}_N$ such that $(m_N^+(\sigma), m_N^-(\sigma)) = (m^+, m^-)$. As we have seen, $P_t[m, \tilde{m}]$ does not depend on the particular σ that satisfies (2.34) since this is true for the infinitesimal generator.

Moreover the forward equation has the form: for all real valued function f on $\mathscr{M}_N$

$$\frac{\partial}{\partial t} \sum_{\tilde{m} \in \mathscr{M}_N} P_t^N[m, \tilde{m}]\ f(\tilde{m}) = \sum_{\tilde{m} \in \mathscr{M}_N} P_t^N[m, \tilde{m}](\mathscr{L}_N f)(\tilde{m}) \tag{2.35}$$

and the Markov process $m(t) \equiv (m_N^+(t), m_N^-(t))$, starting $\hat{m} \in \mathcal{M}_N$ is completely determined by

$$P^N[m_N(t_1) \in \Gamma_1, ..., m_N(t_n) \in \Gamma_n \mid m_N(0) = \hat{m}]$$
$$= \sum_{m(1) \in \Gamma_1} \cdots \sum_{m(n) \in \Gamma_n} P_{t_1}^N[\hat{m}, m(1)]$$
$$\times P_{t_2 - t_1}^N[m(1), m(2)] \cdots P_{t_n - t_{n-1}}^N[m(n-1), m(n)] \tag{2.36}$$

for $0 \leqslant t_1 \leqslant t_2 \cdots \leqslant t_n$.

The following subsets that are discrete versions of the basins of attraction of the absolute minima m_1 are of importance for the study of the metastability:

$$T_1^N = \mathcal{M}_N \cap \{m^+ + m^- \geqslant 0\} \tag{2.37}$$

and

$$\bar{T}_1^N = T_1^N \cup \left\{\left\{m^+ + m^- \geqslant -\frac{3}{N}\right\} \cap \mathcal{M}_N\right\} \tag{2.38}$$

at last

$$\partial T_1^N = \left\{\left\{0 \geqslant m^+ = m^- \geqslant -\frac{3}{N}\right\} \cap \mathcal{M}_N\right\} \tag{2.39}$$

and define $\bar{T}_2^N$ analogously.

Since $\mathcal{L}_N$ is negative it is natural to consider $-\mathcal{L}_N$, and notice that 0 is an eigenvalue of $\mathcal{L}_N$ and the corresponding eigenvector is the constant function that we can take equal to 1 everywhere. More generally, we consider the spectral decomposition of $-\mathcal{L}_N$ that is we consider the set of eigenvalues $\Lambda_i \equiv \Lambda_i^N$ for $i = 0, ..., |\mathcal{M}_N|$ ordered in such a way that $0 = \Lambda_0^N \leqslant \Lambda_1^N \leqslant \cdots \leqslant \Lambda_{|\mathcal{M}_N|}^N$. We normalize the eigenvectors φ_i^N is such a way that

$$\mathcal{G}_N(\varphi_i^N \varphi_j^N) = \mathbb{1}_i(j) \tag{2.40}$$

Since we are interested in the limit $N \nearrow \infty$ we make the convention that $\Lambda_i^N = \infty$ if $i > |\mathcal{M}_N|$.

Let us call $\mathcal{L}_N^K$, the infinitesimal generator of the proccess starting in T_1^N and killed at the time $\tau_N = \inf\{t > 0 : m_N(t) \in \partial T_1^N\}$. $\mathcal{L}_N^K$ is the operator on $L^2(\bar{T}_1^N, \mathcal{G}_N^1)$ where

$$\mathcal{G}_N^1(\Psi) = \frac{Z_N}{Z_N^1} \mathcal{G}_N(\Psi \mathbb{1}_{\bar{T}_1^N}) \tag{2.41}$$

and

$$Z_N^1 \equiv \sum_{m \in \bar{T}_1^N} \sum_{\sigma \in \Sigma_N} \exp(-\beta H_N(\sigma)) \mathbb{1}_{\{m_n(\sigma)=m\}} \tag{2.42}$$

whose Dirichlet form is

$$\mathscr{E}_N^1(\Psi, \Psi) \equiv \frac{Z_N}{Z_N^1} \mathscr{E}_N(\Psi \mathbb{1}_{\bar{T}_1^N}, \Psi \mathbb{1}_{\bar{T}_1^N}) \tag{2.43}$$

and the domain of the Dirichlet form is $\{\phi: \bar{T}_1^N \to \mathbb{R}; \phi(m)=0, m \in \partial T_1^N\}$ that is Dirichlet boundary conditions. The eigenvectors of $-\mathscr{L}_N^K$ will be denoted by $\varphi_i^{N,K}$ for $i \geqslant 1$ with the convention that $\varphi_1^{N,K} > 0$ and normalized in such a way that

$$\mathscr{G}_N^1(\varphi_i^{N,K} \varphi_j^{N,K}) = \mathbb{1}_i(j) \tag{2.44}$$

The corresponding eigenvalues will be denoted $\Lambda_i^{N,K}$ for $i \geqslant 1$, with the convention that $\Lambda_i^{N,K} = \infty$ if i is larger than the rank of the matrix $\mathscr{L}_N^K$.

Let us call $\mathscr{L}_N^R$, the infinitesimal generator of the process starting in T_1^N and reflected on $\bar{T}_1^N$, that is we supress all the jumps that arrive outside $\bar{T}_1^N$. $\mathscr{L}_N^R$ is the operator on $L^2(\bar{T}_1^N, \mathscr{G}_N^1)$ whose Dirichlet form is $\mathscr{E}_N^1(\Psi, \Psi)$ defined in (2.43) with domain $\{\phi: \bar{T}_1^N \to \mathbb{R}\}$, that is Neuman boundary conditions. The eigenvalues of $-\mathscr{L}_N^R$ will be denoted by $\Lambda_i^{N,R}$ for $i \geqslant 0$ with the convention that $\Lambda_i^{N,R} = \infty$ if i is larger than the rank of the matrix $\mathscr{L}_N^R$. The eigenvectors will be denoted by $\varphi_i^{N,R}$, for $i \geqslant 0$ and normalized in such a way that

$$\mathscr{G}_N^1(\varphi_i^{N,R} \varphi_j^{N,R}) = \mathbb{1}_i(j) \tag{2.45}$$

also $\varphi_0^{N,R} \equiv 1$ and $\Lambda_0^{N,R} = 0$.

Using the spectral decomposition, we have

$$\begin{aligned} E_{m_N}[\Psi(m_N(t))] &= (e^{t\mathscr{L}_N}\Psi)(m_N) \\ &= \sum_{i \geqslant 0} \varphi_i^N(m_N)\, e^{-\Lambda_i^N t} \mathscr{G}_N(\Psi \varphi_i) \end{aligned} \tag{2.46}$$

Moreover, using the fact that the process $m_N(t)$ before τ_N coincides with the killed process, we get

$$\begin{aligned} P_{m_N}[\tau_N > t] &= (e^{t\mathscr{L}_N^K}\mathbb{1})(m_N) \\ &= \sum_{i \geqslant 1} \varphi_i^{N,K}(m_N)\, e^{-t\Lambda_i^{N,K}} \mathscr{G}_N^1(\varphi_i^{N,K}) \end{aligned} \tag{2.47}$$

Now we can state our main results. The first result is an rough asymptotic estimate of the exit time of the basin of attraction of m_1. An analogous result is true for the case of m_2.

Theorem 2.1. Let $\tau_N = \inf(t>0 : m_N(t) \in \partial \bar{T}_1^N)$ be the hitting time of $\partial \bar{T}_1^N$. Then for all $\delta > 0$, $\mathbb{P}$-almost surely, for all sequences, $m_N \to m$ such that for all N, $m_N \in T_1^N$, $m \in T_1$ and $\mathscr{F}(m) < \mathscr{F}(m_0)$, we have

$$\lim_{N \nearrow \infty} P_{m_N}[e^{\beta N(\Delta\mathscr{F} - \delta)} \leqslant \tau_N \leqslant e^{\beta N(\Delta\mathscr{F} + \delta)}] = 1 \tag{2.48}$$

The second result is the exponentiality of the exit time of $\bar{T}_1^N$, a similar result holds for the exit time of $\bar{T}_2^N$,

Theorem 2.2. Let $\Lambda_1^{N,K} > 0$ be the first eigenvalue of $-\mathscr{L}_N^K$, minus the infinitesimal generator of the process, killed at time τ_N, then $\mathbb{P}$-almost surely, for all sequences, $m_N \to m$ such that for all N, $m_N \in T_1^N$, $m \in T_1$ and $\mathscr{F}(m) < \mathscr{F}(m_0)$, we have, $\forall t > 0$

$$\lim_{N \nearrow \infty} P_{m_N}\left[\tau_N > \frac{t}{\Lambda_1^{N,K}}\right] = e^{-t} \tag{2.49}$$

As we will see, these results are a consequence of

Theorem 2.3. $\mathbb{P}$-almost surely

$$\lim_{N \nearrow \infty} \frac{1}{\beta N} \log \Lambda_1^{N,K} = -\Delta\mathscr{F} \tag{2.50}$$

This suggest that the right time scale to see interesting behaviors is $e^{\alpha\beta N}$ where α is a new parameter. Depending if $\alpha < \Delta\mathscr{F}$ or $\alpha > \Delta\mathscr{F}$ the stochastic process have two different behaviors. In the first case starting in the basin of attraction of m_1, the system relaxes to m_1 and has not enough time to exit this basin of attraction. In the second case the process is at equilibrium. However due to the randomness of the equilibrium Gibbs measure, very interesting phenomena occur. In particular depending if we are interested in results that are true $\mathbb{P}$-almost surely or just in $\mathbb{P}$-Law, the stochastic process has a different behavior. The first simple case is when $\alpha < \Delta\mathscr{F}$.

Theorem 2.4. If $t = e^{\alpha\beta N}$ with $\alpha < \Delta\mathscr{F}$, then $\mathbb{P}$-almost surely, for all continuous real function Ψ on $[-1, +1]^2$, for all sequences, $m_N \to m$ such that for all N, $m_N \in T_1^N$, $m \in T_1$ and $\mathscr{F}(m) < \mathscr{F}(m_0)$, we have

$$\lim_{N \nearrow \infty} |E_{m_N}[\Psi(m_N(t))] - \Psi(m_1)| = 0 \tag{2.51}$$

In other word, the one dimensional marginal of the process on this time scale converges $\mathbb{P}$-almost surely, weakly to a Dirac mass at the minimum m_1.

The other case is when $\alpha > \Delta\mathcal{F}$ the first result is a $\mathbb{P}$-almost surely one.

Theorem 2.5. If $t = e^{\alpha\beta N}$ with $\alpha > \Delta\mathcal{F}$, then $\mathbb{P}$-almost surely, for all continuous real function Ψ on $[-1, +1]^2$, for all $m_N \in \mathcal{M}_N$

$$\lim_{N \nearrow \infty} |E_{m_N}[\Psi(m_N(t))] - \mathcal{G}_N(\Psi)| = 0 \tag{2.52}$$

We emphasize that this result holds $\mathbb{P}$-almost surely, even if, as we will see later the quantities $\mathcal{G}_N(\Psi)$ *does not* converge $\mathbb{P}$-almost surely but just in Law.

The next result is about the convergence of the one dimensional marginal of the process $m_N(t)$, this is the metastable behavior of the system. This is also a $\mathbb{P}$-almost sure result.

Theorem 2.6. $\mathbb{P}$-almost surely, for all bounded continuous real function Ψ on $[-1, +1]^2$, for all $m_N \in \mathcal{M}_N$, such that $m_N \to m$ with $\mathcal{F}(m) < \mathcal{F}(m_0)$, for all $t > 0$

$$\lim_{N \nearrow \infty} \left| E_{m_N}\left[\Psi\left(m_N\left(\frac{t}{\Lambda_1^N}\right)\right)\right] - (e^{-t}\Psi(m_1) + (1 - e^{-t})\, \mathcal{G}_N(\Psi))\right| = 0 \tag{2.53}$$

A rough asymptotic estimates for Λ_1^N is given in the following

Theorem 2.7. $\mathbb{P}$-almost surely

$$\lim_{N \nearrow \infty} \frac{1}{\beta N} \log \Lambda_1^N = -\Delta\mathcal{F} \tag{2.54}$$

This ends the dynamical results, the next two results are static ones, and show that the measure $\mathcal{G}_N$ has not a very nice $\mathbb{P}$-almost sure behavior, however the $\mathbb{P}$-in Law behavior is nicer. Similar fact was observed for differents measures in ref. 4 and 37.

Theorem 2.8. $\mathbb{P}$-almost surely, the set of cluster points of $\mathcal{G}_N$ is the set $\{\lambda(n)\, \delta_{m_1} + (1 - \lambda(n))\, \delta_{m_2} : n \in \mathbb{Z}\}$ where

$$\lambda(n) = \frac{e^{\beta\tau n}}{e^{\beta\tau n} + e^{-\beta\tau n}} \tag{2.55}$$

for some $\tau = \tau(\beta, \theta)$.

The only possible convergence is a convergence in $\mathbb{P}$ Law to a random convex mean of two Dirac measure, namely

Theorem 2.9. In $\mathbb{P}$-Law

$$\lim_{N \nearrow \infty} \mathcal{G}_N = \lambda \delta_{m_1} + (1-\lambda)\, \delta_{m_2} \tag{2.56}$$

where $\mathbb{P}(\lambda = 1) = 1 - \mathbb{P}(\lambda = 0) = \frac{1}{2}$.

3. SPECTRAL PROPERTIES

In this chapter we collect all the spectral properties we need for proving the main theorems.

3.1. Asymptotics for Spectral Gaps and First Eigenvalues

Consider first the operator $-\mathcal{L}_N$, its Dirichlet form is

$$\mathcal{E}_N(\Psi, \Psi) = \frac{1}{2Z_N} \sum_{\tilde{m} : \tilde{m} \sim m} (\Psi(\tilde{m}) - \Psi(m))^2 \, (\tilde{\mathcal{N}}_N(\tilde{m}, m))^{1/2} \times e^{-\beta N/2\, [\mathcal{F}_N(\tilde{m}) + \mathcal{F}_N(m)]} \tag{3.1}$$

It is immediate that the constant vector equal to 1, $\mathbb{1}$, is an eigenvector with eigenvalue 0 and $\mathcal{G}_N(1) = 1$. By using the variational characterization of eigenvalues we have the following variational formula for the first eigenvalue. It is also the *spectral gap*.

$$\Lambda_1^N = \inf_{\Psi} \left\{ \frac{\mathcal{E}_N(\Psi, \Psi)}{\mathcal{G}_N[(\Psi - \mathcal{G}_N(\Psi))^2]} \right\} \tag{3.2}$$

where the infimun is over the set of real functions on $\mathcal{M}_N$. The first result is

Proposition 3.1. $\mathbb{P}$-almost surely

$$\lim_{N \nearrow \infty} -\frac{1}{\beta N} \log \Lambda_1^N = \Delta\mathcal{F} = \mathcal{F}(m_0) - \mathcal{F}(m_1) \tag{3.3}$$

Proof. Upper Bound. We first give an upper bound on Λ_1^N. Using (3.2) it is enough to choose a trial function. After that, we have to find a lower bound for the variance and an upper bound for the Dirichlet form evaluated on this trial function.

We take $\Psi = \mathbb{1}_{T_1^N} - \mathbb{1}_{T_2^N}$ here

$$T_2^N = \mathcal{M}_N \cap \{m^+ + m^- \leqslant 0\} \tag{3.4}$$

we will need also

$$T_1^{o,N} = \mathcal{M}_N \cap \left\{m^+ + m^- \geqslant \frac{3}{N}\right\} \tag{3.5}$$

and

$$T_2^{o,N} = \mathcal{M}_N \cap \left\{m^+ + m^- \leqslant -\frac{3}{N}\right\} \tag{3.6}$$

Note that $T_1^{o,N} \cap T_2^N = \varnothing$ and $T_2^{o,N} \cap T_1^N = \varnothing$. Let us define

$$\phi(m, \tilde{m}) = \mathbb{1}_{T_1^{o,N}}(m)\, \mathbb{1}_{T_2^{o,N}}(\tilde{m}) + \mathbb{1}_{T_2^{o,N}}(m)\, \mathbb{1}_{T_1^{o,N}}(\tilde{m}) \tag{3.7}$$

Then, using an integral to denote a discrete sum over $\mathcal{M}_N$.

$$\begin{aligned} &\mathcal{G}_N[(\Psi - \mathcal{G}_N(\Psi))^2] \\ &\quad = \tfrac{1}{2} \iint (\Psi(m) - \Psi(\tilde{m}))^2\, \mathcal{G}_N(dm)\, \mathcal{G}_N(d\tilde{m}) \\ &\quad \geqslant \tfrac{1}{2} \iint (\Psi(m) - \Psi(\tilde{m}))^2\, \phi(m, \tilde{m})\, \mathcal{G}_N(dm)\, \mathcal{G}_N(d\tilde{m}) \\ &\quad \geqslant 4\mathcal{G}_N(T_1^{o,N})\, \mathcal{G}_N(T_2^{o,N}) \end{aligned} \tag{3.8}$$

where the last steps follows by expanding the square and some easy algebra.

Using the following lemma, that will be proved in the next chapter,

Lemma 3.2. $\mathbb{P}$-almost surely, for $i = 1, 2$

$$\limsup_{N \nearrow \infty} -\frac{1}{N} \log \mathcal{G}_N(T_i^{o,N}) \leqslant 0 \tag{3.9}$$

it is enough to prove that

$$\limsup_{N \nearrow \infty} \frac{1}{N} \log \mathcal{E}_N(\Psi, \Psi) \leqslant \beta \Delta \mathcal{F} \tag{3.10}$$

to get the wanted upper bound. We have

$$\mathscr{E}_N(\Psi, \Psi) = \frac{1}{2Z_N} \sum_{\substack{\tilde{m}, m \\ : \tilde{m} \sim m}} (\Psi(\tilde{m}) - \Psi(m))^2 (\tilde{\mathscr{N}}_N(\tilde{m}, m))^{1/2} e^{-\beta N/2 [\mathscr{F}_N(\tilde{m}) + \mathscr{F}_N(m)]} \tag{3.11}$$

Since $\Psi = \mathbb{1}_{T_1^N} - \mathbb{1}_{T_2^N}$, it is easy to check that

$$(\Psi(m) - \Psi(\tilde{m}))^2 \leqslant 4 \mathbb{1}_{\Delta_N}(m, \tilde{m}) \tag{3.12}$$

where $\Delta_N = \{(\tilde{m}, m) : \tilde{m} \sim m, |m^+ + m^-| \leqslant 3/N\}$ is the $3/N$ neighborhood of the diagonal Δ in $\mathscr{M}_N$, we get

$$\mathscr{E}_N(\Psi, \Psi) \leqslant 6\sqrt{2}\, \frac{N}{Z_N} e^{-\beta N/2 \inf_{\Delta_N}[\mathscr{F}_N(m) + \mathscr{F}_N(\tilde{m})]} \tag{3.13}$$

now we use the following lemma that will be proved in the next chapter,

Lemma 3.3. $\mathbb{P}$-almost surely, for all $m_N \in \mathscr{M}_N$ such that, $\lim_{N \uparrow \infty} m_N = m \in [-1, +1]^2$, $\lim_{n \uparrow \infty} \mathscr{F}_N(m_N) = \mathscr{F}(m)$.

And we get, for all $\delta > 0$ with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\tfrac{1}{2} \inf_{\Delta_N} [\mathscr{F}_N(m) + \mathscr{F}_N(\tilde{m})] \geqslant \inf_{m \in \Delta} \mathscr{F}(m) - \delta = \mathscr{F}(m_0) - \delta \tag{3.14}$$

The point is that the subset of $\mathbb{P}$-probability 1 where the previous estimates is valid does not depends on m nor m_N.

On the other hand by restricting the set of configurations to a suitable neighborhood of m_1 and using the Lemma 3.3, we get

$$Z_N \geqslant e^{-\beta N(\mathscr{F}(m_1) + \delta)} \tag{3.15}$$

Therefore, collecting (3.13), (3.14) and (3.15) we get, for all $\delta > 0$,

$$\mathscr{E}_N(\Psi, \Psi) \leqslant 6\sqrt{2}\, N e^{2\beta N \delta} e^{-\beta N(\mathscr{F}(m_0) - \mathscr{F}(m_1))} \tag{3.16}$$

with $\mathbb{P}$-probability 1, for all but a finite number of indices N. This leads to

$$\limsup_{N \nearrow \infty} \frac{1}{\beta N} \log \Lambda_1^N \leqslant -\Delta \mathscr{F} \tag{3.17}$$

$\mathbb{P}$-almost surely. Note at this point that here and later, we will write estimates in a slighlty incoherent way. That is, errors terms coming from approximating the function $\mathscr{F}_N$ by its continuous limit $\mathscr{F}$ are always bigger than entropy term coming from the number of points in some subset of $\mathscr{M}_N \mathscr{S}_N$ those ones being always some polynomial in N. However, it seems to us that this way of writing makes the arguments easier to understand.

Lower Bound. To get a lower bound on the spectral gap, recalling (3.2), we make an upper bound on the variance, keeping in mind to reconstruct the Dirichlet form. The point is that this has to be done uniformly with respect to the function Ψ, since we want a lower bound on an infimum over such functions Ψ. We start with

$$\operatorname{Var}(\Psi) = \frac{1}{2} \sum_{m, \tilde{m}} \mathscr{G}_N(m)\, \mathscr{G}_N(\tilde{m})(\Psi(m) - \Psi(\tilde{m}))^2 \tag{3.18}$$

Note here that the previous sum is over all pairs in $\mathscr{M}_N^2$, while in the Dirichlet form the sum is only over all *communicating* pairs, so the first step is to express the difference $\Psi(m) - \Psi(\tilde{m})$ in term of communicating pairs. That is to introduce for any pair $m, \tilde{m} \in \mathscr{M}_N^2$ a path $\gamma_{m,\tilde{m}} \equiv \gamma^N_{m,\tilde{m}}$. That is $\gamma_{m,\tilde{m}}\colon [0, 1] \to \mathscr{M}_N$ and $\gamma_{m,\tilde{m}}(0) = m$. This path will be piecewise constant and will jump at the times $t_i = i/(2N)$ for $i \geqslant 1$ to a point in $\mathscr{M}_N$ that communicates with the point $\gamma_{m,\tilde{m}}(t_{i-1})$. The path will be self avoiding. Since it jumps $2N$ times, it is possible to reach any point $\tilde{m}$ starting from any point m. If it is not necessary to use all the $2N$ steps to reach $\tilde{m}$, we stop the path when it reaches $\tilde{m}$ and stay there. If we specify only that, there are a priori a lot of different choices of paths that can be done. The point is that we have a complete freedom to choose those paths and the good choice is related to the specific problem we handle. A bad choice will give just useless estimates. Extra conditions will be imposed on the paths later. What is important is that we can write

$$\Psi(m) - \Psi(\tilde{m}) = \sum_{i=0}^{2N-1} \Psi(\gamma_{m,\tilde{m}}(t_i)) - \Psi(\gamma_{m,\tilde{m}}(t_{i+1})) \tag{3.19}$$

and by convexity

$$(\Psi(m) - \Psi(\tilde{m}))^2 \leqslant (2N) \sum_{i=0}^{2N-1} [\Psi(\gamma_{m,\tilde{m}}(t_i)) - \Psi(\gamma_{m,\tilde{m}}(t_{i+1}))]^2 \tag{3.20}$$

Therefore, we get

$$\operatorname{Var}(\Psi) \leqslant \frac{N}{Z_N^2} \sum_{m,\tilde{m}} e^{-\beta N(\mathscr{F}_N(m) + \mathscr{F}_N(\tilde{m}))} \sum_{i=0}^{2N-1} [\Psi(\gamma_{m,\tilde{m}}(t_i)) - \Psi(\gamma_{m,\tilde{m}}(t_{i+1}))]^2 \tag{3.21}$$

Note that we have not already reconstructed the Dirichlet form, to do it, we use the following simple

$$
\begin{aligned}
1_{\{\gamma_{m,\tilde m}\in\mathcal{M}_N\}}
&\leqslant e^{-\beta N/2\,[\mathscr{F}_N(\gamma_{m,\tilde m}(t_i))+\mathscr{F}_N(\gamma_{m,\tilde m}(t_{i+1}))]}\, e^{+\beta N/2\,[\mathscr{F}_N(\gamma_{m,\tilde m}(t_i))+\mathscr{F}_N(\gamma_{m,\tilde m}(t_{i+1}))]}\\
&\quad\times(\tilde{\mathcal{N}}(\gamma_{m,\tilde m}(t_i)),\gamma_{m,\tilde m}(t_{i+1})))^{1/2}\,(\tilde{\mathcal{N}}(\gamma_{m,\tilde m}(t_i),\gamma_{m,\tilde m}(t_{i+1})))^{-1/2} \qquad (3.22)
\end{aligned}
$$

With the convention that when the last term in the square root is 0, the product of the last two terms is 0, which is compatible with the left hand side of the inequality. Inserting (3.22) and (3.21), we get

$$
\begin{aligned}
\mathrm{Var}(\Psi)\leqslant{}&\frac{2N}{Z_N}\sup_{m,\tilde m,i}\,[e^{+\beta N/2\,[\mathscr{F}_N(\gamma_{m,\tilde m}(t_i))+\mathscr{F}_N(\gamma_{m,\tilde m}(t_{i+1}))-2\mathscr{F}_N(m)-2\mathscr{F}_N(\tilde m)]}\\
&\times(\tilde{\mathcal{N}}(\gamma_{m,\tilde m}(t_i),\gamma_{m,\tilde m}(t_{i+1})))^{-1/2}]\\
&\times\frac{1}{2Z_N}\sum_{m,\tilde m}\sum_{i=0}^{2N-1}(\Psi(\gamma_{m,\tilde m}(t_i))-\Psi(\gamma_{m,\tilde m}(t_{i+1})))^2\\
&\times(\tilde{\mathcal{N}}(\gamma_{m,\tilde m}(t_i),\gamma_{m,\tilde m}(t_{i+1})))^{1/2}\,e^{-\beta N/2\,[\mathscr{F}_N(\gamma_{m,\tilde m}(t_i))+\mathscr{F}_N(\gamma_{m,\tilde m}(t_{i+1}))]}
\end{aligned}
\qquad (3.23)
$$

The last two lines does not exceed

$$
(2N+1)^3\,\mathscr{E}_N(\Psi,\Psi) \qquad (3.24)
$$

where the term $(2N+1)^3$ comes from the number of terms in the triple sum. Now we have reconstructed the Dirichlet form and putting together (3.23) and (3.24), we get

$$
\begin{aligned}
\mathrm{Var}(\Psi)\leqslant{}&\mathscr{E}_N(\Psi,\Psi)\,\frac{2(2N+1)^4}{Z_N}\\
&\times\sup_{m,\tilde m,i}\,[e^{+\beta N/2\,[\mathscr{F}_N(\gamma_{m,\tilde m}(t_i))+\mathscr{F}_N(\gamma_{m,\tilde m}(t_{i+1}))-2\mathscr{F}_N(m)-2\mathscr{F}_N(\tilde m)]}\\
&\times(\tilde{\mathcal{N}}(\gamma_{m,\tilde m}(t_i),\gamma_{m,\tilde m}(t_{i+1})))^{-1/2}]
\end{aligned}
\qquad (3.25)
$$

We can bound from below the partition function by restricting the spin configurations such that $m_N(\sigma)$ are in a suitable neigborhood of m_1. That is, we have for all $\delta>0$, with $\mathbb{P}$-probability 1, for all but a finite of indice N,

$$
Z_N\geqslant e^{-\beta N(\mathscr{F}(m_1)+\delta)} \qquad (3.26)
$$

It remains to estimate the supremum in (3.25). It is at this point that the choice of the paths is becoming crucial. First note that we have by construction

$$\|\gamma_{m,\tilde{m}}(t_i) - \gamma_{m,\tilde{m}}(t_{i+1})\|_1 \leqslant \frac{2}{N} \tag{3.27}$$

where $\|\cdot\|_1$ is the ℓ^1 norm in $\mathscr{M}_N$. Therefore, using Lemma (3.3), we get, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\begin{aligned}(\mathscr{F}_N(\gamma_{m,\tilde{m}}(t_i)) &+ \mathscr{F}_N(\gamma_{m,\tilde{m}}(t_{i+1})) - 2\mathscr{F}_N(m) - 2\mathscr{F}_N(\tilde{m}))\\ &\leqslant 6\delta + 2 \sup_{m,\tilde{m}} \sup_{0\leqslant t\leqslant 1} [\mathscr{F}(\gamma_{m,\tilde{m}}(t) - \mathscr{F}(m) - \mathscr{F}(\tilde{m})]\end{aligned} \tag{3.28}$$

Taking the term coming from the lower bound (3.26) together with the supremum in the right hand side of (3.28), we have to estimate

$$\tilde{\Delta}\mathscr{F} \equiv \sup_{m,\tilde{m}} \sup_{0\leqslant t\leqslant 1} [\mathscr{F}(\gamma_{m,\tilde{m}}(t) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) + \mathscr{F}(m_1)] \tag{3.29}$$

It is precisely here that the choice of the paths is crucial. As the reader can check, different choices will leads to estimate that are true but rather poor. We will say that a path $\gamma_{m,\tilde{m}}(t)$ passing through m', m'' is $\mathscr{F}$-decreasing from m' to m'', if $\mathscr{F}(\gamma_{m,\tilde{m}}(t_1) \leqslant \mathscr{F}(\gamma_{m,\tilde{m}}(t_2)$ for all $t(m') \leqslant t_1 \leqslant t_2 \leqslant t(m'')$ where $\gamma_{m,\tilde{m}}(t(m')) = m'$ and $\gamma_{m,\tilde{m}}(t(m'')) = m''$. The definition of $\mathscr{F}$-increasing being immediate. Let first take as path from m_1 to m_2 the path passing through m_0 which is $\mathscr{F}$-increasing from m_1 to m_0 and $\mathscr{F}$-decreasing from m_0 to m_2. For this path we have

$$\begin{aligned}\sup_{0\leqslant t\leqslant 1} &[\mathscr{F}(\gamma_{m,\tilde{m}}(t)) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) + \mathscr{F}(m_1)]\\ &= \mathscr{F}(m_0) - \mathscr{F}(m_1) = \Delta\mathscr{F} > 0\end{aligned} \tag{3.30}$$

It remains to construct all the other paths in such a way that the previous one is the maximazer. Let us consider the case where m and $\tilde{m}$ are in the basin of attraction of m_1, i.e., are in T_1 (the case of T_2 being similar). We can assume that $\mathscr{F}(m) \geqslant \mathscr{F}(\tilde{m})$, then we choose for the path $\gamma_{m,\tilde{m}}$ a path which is $\mathscr{F}$-decreasing from m to $\tilde{m}$. With this choice, we get

$$\sup_{0\leqslant t\leqslant 1} [\mathscr{F}(\gamma_{m,\tilde{m}}(t)) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) + \mathscr{F}(m_1)] = -\mathscr{F}(\tilde{m}) + \mathscr{F}(m_1) \leqslant 0 \tag{3.31}$$

It remains to consider the case where $m \in T_1$ and $\tilde{m} \in T_2$, we choose a path which is $\mathscr{F}$-decreasing from m to m_1 (and stay in T_1), then the path is $\mathscr{F}$-increasing from m_1 to m_0, then it is $\mathscr{F}$-decreasing from m_0 to m_2 and is in T_2, at last it is $\mathscr{F}$-increasing from m_2 to $\tilde{m}$ and stay in T_2. Consider first the case where $\sup_{0 \leqslant t \leqslant 1} \mathscr{F}(\gamma_{m,\tilde{m}}(t)) \geqslant \mathscr{F}(m_0)$, we can assume that $\mathscr{F}(m) \geqslant \mathscr{F}(\tilde{m})$, the other case being similar. Then we have $\sup_{0 \leqslant t \leqslant 1} \mathscr{F}(\gamma_{m,\tilde{m}}(t)) - \mathscr{F}(m) = 0$ by construction, therefore

$$\sup_{0 \leqslant t \leqslant 1} [\mathscr{F}(\gamma_{m,\tilde{m}}(t)) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) + \mathscr{F}(m_1)] = -\mathscr{F}(\tilde{m}) + \mathscr{F}(m_1) \leqslant 0 \tag{3.32}$$

Now let us consider the case where $\sup_{0 \leqslant t \leqslant 1} \mathscr{F}(\gamma_{m,\tilde{m}}(t)) \leqslant \mathscr{F}(m_0)$, then, by construction, we have $\sup_{0 \leqslant t \leqslant 1} \mathscr{F}(\gamma_{m,\tilde{m}}(t)) = \mathscr{F}(m_0)$, and we get

$$\begin{aligned}\sup_{0 \leqslant t \leqslant 1} & [\mathscr{F}(\gamma_{m,\tilde{m}}(t)) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) + \mathscr{F}(m_1)] \\ &= \mathscr{F}(m_0) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) + \mathscr{F}(m_1) \\ &\leqslant \mathscr{F}(m_0) - \mathscr{F}(m_1)\end{aligned} \tag{3.33}$$

Collecting what we have done, we get, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite of indices N,

$$\frac{\mathscr{E}_N(\Psi, \Psi)}{\mathrm{Var}(\Psi)} \geqslant (2N+1)^{-4} e^{-\beta N 4\delta} e^{-\beta N \Delta \mathscr{F}} \tag{3.34}$$

and this end the proof of the Proposition (3.1).

We consider now the operator $-\mathscr{L}_N^R$, that is minus the generator of the process reflected on $\bar{T}_1^N$, this correspond to Neuman boundary conditions. It is immediate that the constant vector equal $\mathbb{1}_{T_1^N}$ is an eigenvector with eigenvalue 0. Calling $\mathscr{G}_N^1$, the *normalized* Gibbs measure restricted to $\bar{T}_1^N$. We have $\mathscr{G}_N^1(\mathbb{1}_{T_1^N}) = 1$. The spectral gap is given by

$$\Lambda_1^{N,R} = \inf_{\Psi} \left\{ \frac{\mathscr{E}_N^1(\Psi, \Psi)}{\mathscr{G}_N^1[(\Psi - \mathscr{G}_N^1(\Psi))^2]} \right\} \tag{3.35}$$

where the infimum is over the set of real function on $\bar{T}_1^N$. Here we have

Proposition 3.4. $\mathbb{P}$-almost surely

$$\lim_{N \nearrow \infty} -\frac{1}{\beta N} \log \Lambda_1^{N,R} = 0 \tag{3.36}$$

Proof. The proof is completely similar to the one of the previous Proposition. We just mention here the differences. First of all, we have, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\frac{\mathscr{E}_N^1(\Psi, \Psi)}{\mathscr{G}_N^1[(\Psi - \mathscr{G}_N^1(\Psi))^2]} \geqslant (2N+1)^{-4} e^{-4\beta N\delta} e^{-\beta N[\sup_{m, \tilde{m} \in T_1^N} \sup_{0 \leqslant t \leqslant 1} (\mathscr{F}(\gamma_{m, \tilde{m}}(t)) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) + \mathscr{F}(m_1))]} \tag{3.37}$$

Here we choose the paths $\gamma_{m, \tilde{m}}(t)$ such that $\gamma_{m, \tilde{m}}(t) \in \bar{T}_1$ and $\mathscr{F}(\gamma_{m, \tilde{m}}(t)) \leqslant \mathscr{F}(m) \vee \mathscr{F}(\tilde{m})$ then we have

$$\begin{aligned} \sup_{0 \leqslant t \leqslant 1} \mathscr{F}(\gamma_{m, \tilde{m}}(t)) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) &\leqslant \mathscr{F}(m) \vee \mathscr{F}(\tilde{m}) - \mathscr{F}(m) - \mathscr{F}(\tilde{m}) \\ &\leqslant -\mathscr{F}(m_1) \end{aligned} \tag{3.38}$$

which implies that for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N

$$\Lambda_1^{N, R} \geqslant (2N+1)^{-4} e^{-4\beta N\delta} \tag{3.39}$$

from which we get, $\mathbb{P}$-almost surely

$$\limsup_{N \nearrow \infty} -\frac{1}{\beta N} \log \Lambda_1^{N, R} \leqslant 0 \tag{3.40}$$

To get an upper bound for $\Lambda_1^{N, R}$, we consider the trial function $\Psi_\rho = \mathbb{1}_{B_\rho(m_1)} - \mathbb{1}_{B_\rho^c(m_1)}$ where $B_\rho(m_1)$ is a ball of radius ρ centered at m_1, for some ρ to be chosen later. We have, for all $\rho > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\frac{\mathscr{E}_N^1(\Psi_\rho, \Psi_\rho)}{\mathscr{G}_N^1[(\Psi_\rho - \mathscr{G}_N^1(\Psi_\rho))^2]} \leqslant Z_N^1 \, |\partial B_\rho(m_1)| \, \frac{e^{-\beta N \inf_{m \in \partial B_\rho(m_1)} (\mathscr{F}(m) - \delta)}}{e^{-\beta N(\mathscr{F}(m_1) + \mathscr{F}(\bar{m}) + \delta)}} \tag{3.41}$$

here $\bar{m}$ is such that $\mathscr{F}(\bar{m}) = \inf_{m \in B_\rho(m_1)} (\mathscr{F}(m)$. To get the last bound we have used the fact that in the Dirichlet form, $m \sim \tilde{m}$ implies $\Psi_\rho(m) - \Psi_\rho(\tilde{m}) \neq 0$ only if $m \in B_\rho(m_1)$ and $\tilde{m} \in B_\rho^c(m_1) \cap \bar{T}_1^N$ and this give the factor $|\partial B_\rho(m_1)|$. Recalling that the variance is a sum over all pairs m and $\tilde{m}$, a lower bound for the variance is obtained by restricting the configurations to those ones where $m = \bar{m}$ and $\tilde{m} = m_1$. Now we can use, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$Z_N^1 \leqslant N^2 e^{-\beta N[\mathscr{F}(m_1) - \delta]} \tag{3.42}$$

Therefore, collecting (3.41) and (3.42), we get, $\mathbb{P}$-almost surely

$$\liminf_{N \nearrow \infty} -\frac{1}{\beta N} \log \Lambda_1^{N, R} \geqslant 0 \tag{3.43}$$

Now, (3.40) and (3.43) implies (3.36).

We consider the infinitesimal generator of the process killed when it reaches ∂T_1^N This correspond to Dirichlet boundary conditions. Here the first eigenvalue of $-\mathscr{L}_N^K$ is given by the variational formula

$$\Lambda_1^{N, K} = \inf_{\Psi|_{\partial T_1^N} = 0} \frac{\mathscr{E}_N^1(\Psi, \Psi)}{\mathscr{G}_N^1((\Psi)^2)} \tag{3.44}$$

Proposition 3.5. $\mathbb{P}$-almost surely

$$\lim_{N \nearrow \infty} -\frac{1}{\beta N} \log \Lambda_1^{N, K} = \Delta\mathscr{F} \equiv \mathscr{F}(m_0) - \mathscr{F}(m_1) \tag{3.45}$$

Proof. Here also the proof is similar to the one of the Proposition 3.1. For the lower bound we make a path argument as before. Taking into account that the infimum in (3.44) is over the set of real functions such that $\Psi|_{\partial T_1^N} = 0$, we take a path $\gamma_{m, \tilde{m}}(t) \in \bar{T}_1^N$ such that $m = \gamma_{m, \tilde{m}}(0)$, $\tilde{m} = \gamma_{m, \tilde{m}}(1) \in \partial T_1^N$ with $|\gamma_{m, \tilde{m}}(1) - m_0| \leqslant 4/N$ since $\Psi(\gamma_{m, \tilde{m}}(1)) = 0$ we have

$$\Psi(m) = \sum_{i=1}^{2N} \Psi(\gamma_{m, \tilde{m}}(t_i)) - \Psi(\gamma_{m, \tilde{m}}(t_{i+1})) \tag{3.46}$$

Making exactly the same estimates as before, we get, for all $\delta > 0$, $\mathbb{P}$-almost surely, for all but a finite number of indices N

$$\Lambda_1^{N, K} \geqslant e^{-\beta\delta N} e^{-\beta N[\sup_{m \in \bar{T}_1^N} \inf_{\tilde{m} \in \partial T_1^N} \sup_{0 \leqslant t \leqslant 1} \mathscr{F}(\gamma_{m, \tilde{m}}(t)) - \mathscr{F}(m)]} \tag{3.47}$$

where the infimum over $\tilde{m} \in \partial T_1^N$ comes from the fact that we can choose the final point as we want, in particular the one that minimizes the quantity that comes into play.

We choose for the paths, an $\mathscr{F}$-increasing ones from m to m_0 if $\mathscr{F}(m) \leqslant \mathscr{F}(m_0)$ and an $\mathscr{F}$-decreasing ones from m to m_0 if $\mathscr{F}(m) \geqslant \mathscr{F}(m_0)$. It is easy to check that we get after minimizing on the point $\tilde{m} \in \partial T_1^N$, using $\mathscr{F}(m) \geqslant \mathscr{F}(m_1)$

$$\sup_{m \in \bar{T}_1^N} \inf_{\tilde{m} \in \partial T_1^N} \sup_{0 \leqslant t \leqslant 1} \mathscr{F}(\gamma_{m, \tilde{m}}(t)) - \mathscr{F}(m) \leqslant \mathscr{F}(m_0) - \mathscr{F}(m_1) \tag{3.48}$$

Collecting (3.47) and (3.48) we get, $\mathbb{P}$-almost surely

$$\limsup_{N \nearrow \infty} -\frac{1}{\beta N} \log \Lambda_1^{N,K} \leqslant \Delta \mathscr{F} = \mathscr{F}(m_0) - \mathscr{F}(m_1) \tag{3.49}$$

To get an upper bound for $\Lambda_1^{N,K}$, we take as trial function $\Psi \equiv \mathbb{1}_{\{m \notin \partial T_1^N\}}$. We make first an estimate from below of the $L^2(\bar{T}_1^N, \mathscr{G}_N^1)$ norm of this trial function. We have

$$\begin{aligned} \sum_{m \in \bar{T}_1^N} (\Psi(m))^2 \mathscr{G}_N^1(m) &= \sum_{m \in \bar{T}_1^N \setminus \partial T_1^N} \mathscr{G}_N^1(m) \\ &\geqslant \frac{e^{-\beta N[\inf_{m_N \in T_1^N \setminus \partial T_1^N} \mathscr{F}_N(m_N)]}}{Z_N^1} \end{aligned} \tag{3.50}$$

But

$$Z_1^N \leqslant (2N+1)^2 \, e^{-\beta N[\inf_{m_N \in \bar{T}_1^N} \mathscr{F}_N(m_N)]} \tag{3.51}$$

since for all $\delta > 0$, $\mathbb{P}$-almost surely for all but a finite number of indices N

$$\inf_{m_N \in T_1^N \setminus \partial T_1^N} \mathscr{F}_N(m_N) - \inf_{m_N \in \bar{T}_1^N} \mathscr{F}_N(m_N) \leqslant 2\delta \tag{3.52}$$

we get

$$\sum_{m \in \bar{T}_1^N \setminus \partial T_1^N} \mathscr{G}_N^1(m) \geqslant \frac{e^{-2\beta\delta N}}{(2N+1)^2} \tag{3.53}$$

Now we make an estimate from above of the Dirichlet form. With our choice of Ψ, we have $\Psi(m) - \Psi(\tilde{m}) = 0$ except if $m \in \partial T_1^N$ and $\tilde{m} \in \bar{T}_1^N \setminus \partial T_1^N$ or the same with m exchanged with $\tilde{m}$, let us call $\tilde{\Delta}_N$ this set, there $(\Psi(m) - \Psi(\tilde{m}))^2 = 1$, therefore for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N

$$\begin{aligned} Z_N^1 \mathscr{E}_N^1(\Psi, \Psi) &\leqslant \frac{(2N+1)^2}{2} \sup_{\substack{m, \tilde{m} \in \tilde{\Delta}_N \\ m \sim \tilde{m}}} e^{-\beta N/2 \, [\mathscr{F}_N(m) + \mathscr{F}_N(\tilde{m})]} \\ &\leqslant \frac{(2N+1)^2}{2} e^{-\beta N[\mathscr{F}(m_0) - \delta]} \end{aligned} \tag{3.54}$$

since we have, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N

$$Z_N^1 \geqslant e^{-\beta N[\mathscr{F}(m_1) + \delta]} \tag{3.55}$$

Therefore, using (3.54) and (3.55), we get for all δ, with $\mathbb{P}$-probability 1, for all but a finite number of indices N

$$\mathscr{E}_N^1(\Psi, \Psi) \leqslant \frac{(2N+1)^2}{2} e^{-\beta N[\mathscr{F}(m_0) - \mathscr{F}(m_1) - 2\delta]} \tag{3.56}$$

which implies, $\mathbb{P}$-almost surely,

$$\liminf_{N \nearrow \infty} -\frac{1}{\beta N} \log \Lambda_1^{N,K} \geqslant \Delta\mathscr{F} \leqslant \mathscr{F}(m_0) - \mathscr{F}(m_1) \tag{3.57}$$

Collecting (3.49) and (3.57) we get (3.44) and this end the proof of the Proposition.

3.2. Asymptotics for the Second Eigenvalues

We consider the second eigenvalues of $-\mathscr{L}_N^K$ and $-\mathscr{L}_N$. Using the minimax characterization of the eigenvalues, we have

$$\begin{aligned} \Lambda_2^{N,K} &= \sup_{\varphi : \varphi|_{\partial T_1^N} = 0} \inf_{\substack{\Psi : \mathscr{G}_1^N(\varphi\Psi) = 0 \\ \Psi|_{\partial T_1^N} = 0}} \frac{\mathscr{E}_N^1(\Psi, \Psi)}{\mathscr{G}_N^1(\Psi^2)} \\ &\geqslant \inf_{\substack{\Psi : \mathscr{G}_1^N(\Psi) = 0 \\ \Psi|_{\partial T_1^N} = 0}} \frac{\mathscr{E}_N^1(\Psi, \Psi)}{\mathscr{G}_N^1(\Psi^2)} \geqslant \inf_{\Psi : \mathscr{G}_1^N(\Psi) = 0} \frac{\mathscr{E}_N^1(\Psi, \Psi)}{\mathscr{G}_N^1(\Psi^2)} \end{aligned} \tag{3.58}$$

where the first inequality follows by choosing as function $\varphi(m) = \mathbb{1}_{m \notin \partial T_1^N}$. But the right hand side is just the variational characterization of the minimal eigenvalue of minus the infinitesimal generator of the reflected process on $\bar{T}_1^N$, using (3.39) we get, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\Lambda_2^{N,K} \geqslant (2N+1)^{-4} e^{-4\beta N\delta} \tag{3.59}$$

We consider now the second eigenvalue of $-\mathscr{L}_N$. Again by the variational characterization of eigenvalues, we have

$$\Lambda_2^N = \sup_{\mathscr{V}} \inf_{\Psi \perp \mathscr{V}} \frac{\mathscr{E}_N(\Psi, \Psi)}{\mathscr{G}_N(\Psi^2)} \tag{3.60}$$

Here the supremum is over $\mathscr{V}$ that are two dimensional subspaces of $L^2(\mathscr{M}_N, \mathscr{G}_N)$. Taking $\mathscr{V} = \text{span}(\mathbb{1}_{\bar{T}_1^N}, \mathbb{1}_{\bar{T}_2^N})$, the two dimensional subspace generated by those two vectors, $\Psi \perp \mathscr{V}$ means just

$$\mathscr{G}_N^1(\Psi) = \mathscr{G}_N^2(\Psi) = 0 \tag{3.61}$$

here $\mathscr{G}_N^2$ is the (normalized) Gibbs measure restricted on $\bar{T}_2^N$ that is for any Ψ real valued function on $\bar{T}_2^N$,

$$\mathscr{G}_N^2(\Psi) = \frac{Z_N}{Z_N^2} \mathscr{G}_N(\Psi \mathbb{1}_{\bar{T}_2^N}) \tag{3.62}$$

with

$$Z_N^2 \equiv \sum_{m \in \bar{T}_2^N} \sum_{\sigma \in \Sigma_N} \exp(-\beta H_N(\sigma)) \mathbb{1}_{\{m_n(\sigma) = m\}} \tag{3.63}$$

We want a lower bound for Λ_2^N, that is we want a upper bound of $\mathscr{G}_N(\Psi^2)$ in term of the Dirichlet form. We have

$$\mathscr{G}_N(\Psi^2) \leqslant \mathscr{G}_N^1(\Psi^2) \frac{Z_N^1}{Z_N} + \mathscr{G}_N^2(\Psi^2) \frac{Z_N^2}{Z_N} \tag{3.64}$$

Since there are some double counting in the diagonal. Using the orthogonality condition (3.61) we get

$$\mathscr{G}_N(\Psi^2) = \mathscr{G}_N^1[(\Psi - \mathscr{G}_N^1(\Psi))^2] \frac{Z_N^1}{Z_N} + \mathscr{G}_N^2[(\Psi^2 - \mathscr{G}_N^2(\Psi))^2] \frac{Z_N^2}{Z_N} \tag{3.65}$$

Now the crucial observation is that for $i = 1$ or $i = 2$

$$\mathscr{G}_N^i[(\Psi - \mathscr{G}_N^i(\Psi))^2] \leqslant \frac{\mathscr{E}_N^i(\Psi, \Psi)}{\Lambda_1^{N,R,i}} \tag{3.66}$$

where $\Lambda_1^{N,R,i}$ is the first eigenvalue of minus the infinitesimal generator of the process reflected on $\bar{T}_1^N$ for $i = 1$ and reflected on $\bar{T}_2^N$ for $i = 2$. It follows from (3.39), and corresponding modifications for the case $i = 2$, that for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\Lambda_1^{N,R,i} \geqslant (2N+1)^{-4} e^{-4\beta N\delta} \tag{3.67}$$

from which we get

$$\mathcal{G}_N(\Psi^2) \leqslant \frac{Z_N^1}{Z_N} \frac{\mathcal{E}_N^1(\Psi, \Psi)}{\Lambda_1^{N,R,1}} + \frac{Z_N^2}{Z_N} \frac{\mathcal{E}_N^2(\Psi, \Psi)}{\Lambda_1^{N,R,2}}$$
$$\leqslant 2\mathcal{E}_N(\Psi, \Psi)\, 2(2N+1)^4\, e^{+4\beta N\delta} \tag{3.68}$$

where we have used the simple fact that

$$\frac{Z_N^i}{Z_N} \mathcal{E}_N^i(\Psi, \Psi) \leqslant \mathcal{E}_N(\Psi, \Psi) \tag{3.69}$$

Therefore, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\Lambda_2^N \geqslant \frac{1}{2(2N+1)^4} e^{-4\beta N\delta} \tag{3.70}$$

3.3. Estimates of the First Eigenvectors $\varphi_1^{N,K}$

Let $\varphi_1^{N,K}$ be the first eigenvector of the infinitesimal generator of the process killed when it reaches ∂T_1^N. Taking into account of (2.47), it is clear that to get (2.49) we need to prove that $\varphi_1^{N,K}$ converges to 1 in the sup norm. That is we have to find an $L^\infty(\mathcal{M}_N, \mathcal{G}_N)$, $L^2(\mathcal{M}_N, \mathcal{G}_N)$ estimate. We already know that $\varphi_1^{N,K}(m) > 0$ and

$$\mathcal{G}_N^1[(\varphi_1^{N,K})^2] = 1 \tag{3.71}$$

the result of this subsection is the following proposition

Proposition 3.6. With $\mathbb{P}$-probability 1, for all $\bar{m}_N \in T_1^N$ such that $\lim_{N \nearrow \infty} \bar{m}_N = \bar{m}$ and $\mathcal{F}(\bar{m}) < \mathcal{F}(m_0)$,

$$\lim_{N \nearrow \infty} \varphi_1^{N,K}(\bar{m}_N) = 1 \tag{3.72}$$

Proof. Let us first remark that, with $\mathbb{P}$-probability 1,

$$\lim_{N \nearrow \infty} \mathcal{G}_N^1(\varphi_1^{N,K}) = 1 \tag{3.73}$$

This follows from (3.71) using the fact that,

$$1 - (\mathcal{G}_N^1[\varphi_1^{N,K}])^2 = \mathcal{G}_N^1[(\varphi_1^{N,K} - \mathcal{G}_N^1(\varphi_1^{N,K}))^2] \equiv \mathrm{Var}_N^1[\varphi_1^{N,K}] \tag{3.74}$$

Now recalling that for all Ψ, see (3.37) and (3.39),

$$\frac{\mathscr{E}_N^1(\Psi, \Psi)}{\mathrm{Var}_N^1(\Psi)} \geqslant \Lambda_1^{N,R} \geqslant (2N+1)^{-4} e^{-4\beta N\delta} \tag{3.75}$$

and moreover, since $\mathscr{E}_N^1(\varphi_1^{N,K}, \varphi_1^{N,K}) = \Lambda_1^{N,K}$ we get

$$\mathscr{G}_N^1[(\varphi_1^{N,K} - \mathscr{G}_N^1(\varphi_1^{N,K}))^2] \leqslant \frac{\Lambda_1^{N,K}}{\Lambda_1^{N,R}} \leqslant (2N+1)^4 e^{+6\beta N\delta} e^{-\beta N\Delta\mathscr{F}} \tag{3.76}$$

where we have used (3.45). On the other hand by the Schwarz inequality

$$(\mathscr{G}_N^1[\varphi_1^{N,K}])^2 \leqslant \mathscr{G}_N^1[(\varphi_1^{N,K})^2] = 1 \tag{3.77}$$

therefore, collecting (3.76) and (3.77) we get (3.73).

To get an uniform estimate, using (3.73), we start with

$$[\varphi_1^{N,K}(\bar{m}_N) - \mathscr{G}_N^1(\varphi_1^{N,K})]^2 \leqslant \sum_{m_N \in T_1^N} \mathscr{G}_N^1(m_N)[\varphi_1^{N,K}(\bar{m}_N) - \varphi_1^{N,K}(m_N)]^2 \tag{3.78}$$

by convexity. We make a path argument as before and write

$$\varphi_1^{N,K}(\bar{m}_N) - \varphi_1^{N,K}(m_N) = \sum_{i=0}^{2N-1} \varphi_1^{N,K}(\gamma_{\bar{m}_N, m_N}(t_i)) - \varphi_1^{N,K}(\gamma_{\bar{m}_N, m_N}(t_{i+1})) \tag{3.79}$$

Now, again by convexity, using (3.22), we get

$$\begin{aligned}
&[\varphi_1^{N,K}(\bar{m}_N) - \mathscr{G}_N^1(\varphi_1^{N,K})]^2 \\
&\quad\leqslant \frac{2N+1}{Z_N^1} \sum_{m_N \in T_1^N} e^{-\beta N \mathscr{F}_N(m_N)} \\
&\qquad\times \sum_{i=0}^{2N-1} [\varphi_1^{N,K}(\gamma_{\bar{m}_N, m_N}(t_i)) - \varphi_1^{N,K}(\gamma_{\bar{m}_N, m_N}(t_{i+1}))]^2 \\
&\quad\leqslant 2(2N) \sup_{m_N \in T_1^N,\, t_i} [e^{+\beta N/2\, [\mathscr{F}_N(\gamma_{\bar{m}_N, m_N}(t_i)) + \mathscr{F}_N(\gamma_{\bar{m}_N, m_N}(t_{i+1})) - 2\mathscr{F}_N(m_N)]} \\
&\qquad\times (\tilde{\mathscr{N}}(\gamma_{\bar{m}_N, m_N}(t_i), \gamma_{\bar{m}_N, m_N}(t_{i+1})))^{-1/2}] \\
&\qquad\times \frac{1}{2Z_N^1} \sum_{m_N \in T_1^N} \sum_{i=0}^{2N} [\varphi_1^{N,K}(\gamma_{\bar{m}_N, m_N}(t_i)) - \varphi_1^{N,K}(\gamma_{\bar{m}_N, m_N}(t_{i+1}))]^2 \\
&\qquad\times (\tilde{\mathscr{N}}(\gamma_{\bar{m}_N, m_N}(t_i), \gamma_{\bar{m}_N, m_N}(t_{i+1})))^{1/2} \\
&\qquad\times e^{-\beta N/2\, [\mathscr{F}_N(\gamma_{\bar{m}_N, m_N}(t_i)) + \mathscr{F}_N(\gamma_{\bar{m}_N, m_N}(t_{i+1}))]}
\end{aligned} \tag{3.80}$$

For all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N, the last line is bounded by

$$
\begin{aligned}
(2N+1)^3 \mathscr{E}_N^1(\varphi_1^{N,K}, \varphi_1^{N,K}) &= (2N+1)^3 \Lambda_1^{N,K} \\
&\leqslant (2N+1)^5 e^{-\beta N(\Delta \mathscr{F} - 2\delta)}
\end{aligned}
\tag{3.81}
$$

where we have used the fact that $\varphi_1^{N,K}$ is an eigenvector and the estimate (3.56) for the corresponding eigenvalue. Therefore we get

$$
\begin{aligned}
&[\varphi_1^{N,K}(\bar{m}_N) - \mathscr{G}_N^1(\varphi_1^{N,K})]^2 \\
&\quad \leqslant 2(2N+1)^5 e^{-\beta N(\Delta \mathscr{F} - 2\delta)} \\
&\quad\quad \times \sup_{m_N \in T_1^N,\, t_i} e^{+\beta N/2\, [\mathscr{F}(\gamma_{\bar{m}_N, m_N}(t_i)) + \mathscr{F}(\gamma_{\bar{m}, m_N}(t_{i+1})) - 2\mathscr{F}(m_N) - 4\delta]}
\end{aligned}
\tag{3.82}
$$

It remains to make a good choice of the paths $\gamma_{\bar{m}_N, m_N}(t_i)$. If m_N is such that $\mathscr{F}(m_N) \leqslant \mathscr{F}(\bar{m}_N)$ we choose for $\gamma_{\bar{m}_N, m_N}(t_i)$ a $\mathscr{F}$-decreasing path from $\bar{m}_N$ to m_N. Therefore, we get, in this case, for some $\eta > 0$, if N is large enough

$$
\begin{aligned}
\mathscr{F}(\gamma_{\bar{m}_N, m_N}(t_i)) + \mathscr{F}(\gamma_{\bar{m}, m_N}(t_{i+1})) - 2\mathscr{F}(m_N) \\
\leqslant 2(\mathscr{F}(\bar{m}_N) - \mathscr{F}(m_N) + \delta) \\
\leqslant 2(\mathscr{F}(m_0) - \mathscr{F}(m_1)) + 6\delta - 2\eta
\end{aligned}
$$

where we have used at the last step that $\mathscr{F}(\bar{m}) < \mathscr{F}(m_0)$ to get the -2η.

If m_N is such that $\mathscr{F}(m_N) \geqslant \mathscr{F}(\bar{m}_N)$ we choose for $\gamma_{\bar{m}_N, m_N}(t_i)$ a $\mathscr{F}$-increasing path from $\bar{m}_N$ to m_N. Therefore, we get, in that case, if N is large enough

$$
\begin{aligned}
\mathscr{F}(\gamma_{\bar{m}_N, m_N}(t_i)) + \mathscr{F}(\gamma_{\bar{m}, m_N}(t_{i+1})) - 2\mathscr{F}(m_N) &\leqslant 2(\mathscr{F}(m_N) - \mathscr{F}(m_N) + \delta) \\
&\leqslant 2\delta
\end{aligned}
\tag{3.84}
$$

Inserting (3.83) and (3.84) in (3.82), for all $\eta > 0$, choosing $\delta \leqslant \eta/12$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N, we have

$$
[\varphi_1^{N,K}(\bar{m}_N) - \mathscr{G}_N^1(\varphi_1^{N,K})]^2 \leqslant 2(2N+1)^5 e^{-\beta N \eta}
\tag{3.85}
$$

from which, using (3.73), we get (3.72) and this ends the proof of the Proposition 3.6.

3.4. Estimates on the Eigenvectors

In this subsection we give a rough uniform estimate on the $L^\infty(\mathscr{M}_N, \mathscr{G}_N)$ norm of the eigenvectors of the infinitesimal generator of the killed process. This is a general $L^\infty(\mathscr{M}_N, \mathscr{G}_N)$, $L^2(\mathscr{M}_N, \mathscr{G}_N)$ estimates.

Proposition 3.7. For all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N, for all Ψ real valued function on $\mathscr{M}_N$

$$\sup_{m_N \in \mathscr{M}_N} |\Psi(m_N)| \leqslant (2N+1)^2 e^{\beta N[\operatorname{Osc}(\mathscr{F}) + 2\delta]} \|\Psi\|_2 \tag{3.86}$$

where $\|\cdot]\|_2$ is the $L^2(\mathscr{M}_N, \mathscr{G}_N)$ norm and

$$\operatorname{Osc}(\mathscr{F}) = \sup_{m \in [-1, +1]^2} \mathscr{F}(m) - \inf_{m \in [-1, +1]^2} \mathscr{F}(m) \tag{3.87}$$

Proof. First note that, given $m_N \in \mathscr{M}_N$, we have, for all $\delta > 0, \varepsilon > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\begin{aligned} \|\Psi\|_2 &\geqslant \|\mathbb{1}_{B_\rho(m_N)} \Psi\|_2 \\ &\geqslant e^{-\beta N \sup_{m \in B_\rho(m_N)} [\mathscr{F}(m) + \delta]} e^{\beta N[\mathscr{F}(m_1) - \delta]} (2N+1)^{-2} \\ &\quad \times (1 + e^{\beta N \varepsilon^2})^{-1} \inf_{m \in B_\rho(m_N)} |\Psi(m)| \end{aligned} \tag{3.88}$$

This implies

$$\inf_{m \in B_\rho(m_N)} |\Psi(m)| \leqslant (2N+1)^2 (1 + e^{\beta N \varepsilon^2}) e^{\beta N[\operatorname{Osc}(\mathscr{F}) + 2\delta]} \|\Psi\|_2 \tag{3.89}$$

from which we get (3.86) immediately by choosing for m_N the point where the sup in (3.86) is realized and then $\rho \downarrow 0$.

4. STATIC PROPERTIES

In this chapter, we prove all the results we need for the equilibrium measure. We were unable to find all of its in the literature, even if similar computations were done by anyone who work in this subject, in particular in refs. 3, 4, and 37. We will start with regularity properties of the "canonical" free energy. Let us recall, see (2.17), that

$$\mathscr{F}_N(m^+, m^-) = -\frac{1}{2}(m^+ + m^-)^2 - \theta(m^+ - m^-)$$
$$-\frac{1}{\beta N}\log\left(\binom{N^+}{\left(1+\frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}}\binom{N^-}{\left(1+\frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}}\right) \tag{4.1}$$

We use Stirling's formula as given by Robbins,[59] valid for $n \geqslant 1$:

$$n! = n^{n+1/2}e^{-n+\varepsilon_n}\sqrt{2\pi} \tag{4.2}$$

with

$$\frac{1}{12n+1} \leqslant \varepsilon_n \leqslant \frac{1}{12n} \tag{4.3}$$

and we get

$$\binom{n}{(1+x)\frac{n}{2}} = \frac{1}{\sqrt{2\pi[(1-x^2)/4]}}\, e^{-n[I(x)+\mathscr{E}_n(x)]} \tag{4.4}$$

where $I(x)$ is defined in (2.19) and

$$\mathscr{E}_n(x) = \varepsilon_n - \varepsilon_{((1+x)n)/2} - \varepsilon_{((1-x)n)/2} \tag{4.5}$$

Proof of the Lemma 3.3. It is easy and standard that for all $x > 0$, we have

$$\mathbb{P}\left[\left|N_+ - \frac{N}{2}\right| \geqslant x\right] \leqslant 2e^{-(2x^2)/N} \tag{4.6}$$

therefore, for all $p > 0$

$$\mathbb{P}\left[\left|\rho_N^+ - \frac{1}{2}\right| \geqslant \sqrt{\frac{(p+\gamma)\log N}{2N}}\right] \leqslant \frac{2}{N^{p+\gamma}} \tag{4.7}$$

Given $\varepsilon > 1/N$, say smaller than e^{-1}, let us consider $m^+ \in \{\rho_N^+ - (2k)/N,..., \rho_N^+\}$ for $k/N \leqslant \varepsilon$, since we have extended $I(x)$ by 0 for $x > 1$ we have, using the first Borel–Cantelli lemma, for all but a finite number of indices N,

$$\sup_{k\,:\,k/N\leqslant\varepsilon}\left|I\left(\frac{m^+}{\rho_N^+}\right)-I(1)\right|$$
$$=\sup_{k\,:\,k/N\leqslant\varepsilon}\left|\left(1-\frac{k}{N\rho_N^+}\right)\log\left(1-\frac{k}{N\rho_N^+}\right)+\frac{k}{N\rho_N^+}\log\frac{k}{N\rho_N^+}\right|$$
$$\leqslant c\varepsilon\log\frac{1}{\varepsilon} \tag{4.8}$$

for some positive constant c, by choosing $p>2$. On the other hand for all $0<\gamma<1$ and all $x>0$, $y>0$ we have

$$|x\log x-y\log y|\leqslant|x-y|+\frac{1}{\gamma(1-\gamma)\,e}\,|x-y|^{\gamma} \tag{4.9}$$

putting together (4.8) and (4.9) we get, for all $\varepsilon>0$, $0<\gamma<1$, $\mathbb{P}$-almost surely for all but a finite number of indices N

$$\sup_{\substack{(m_N^+,\,m^+)\\ \|m_N^+-m^+\|\leqslant\varepsilon}}\left|I\left(\frac{m_N^+}{\rho_N^+}\right)-I(2m^+)\right|\leqslant c\varepsilon^{\gamma} \tag{4.10}$$

for some positive constant c, choosing $p>3$, since there are no more than $(2N+1)^2$ points in $\mathscr{M}_N$. Now it is easy to check that, if $(m_N^+)/(\rho_N^+)=1-(2k)/(N\rho_N^+)$ and $k\geqslant\varepsilon N$ for some $\varepsilon>0$ then with $\mathbb{P}$-probability 1, for all but a finite number of indices N

$$|\mathscr{E}_N(m_N^+)|\leqslant\frac{c}{N^2\varepsilon} \tag{4.11}$$

To control $\mathscr{E}_N(m_N^+)$ when $1\leqslant k\leqslant\varepsilon N$, it is easier to use another inequality, namely

$$N^+\leqslant\binom{N^+}{N^+-2k}\leqslant\frac{(N^+)^{2k}}{(2k)!}\leqslant c\left(\frac{N^+}{2k}\right)^{2k}e^{2k} \tag{4.12}$$

therefore, if $k\leqslant\varepsilon N$, for some positive constant c, with $\mathbb{P}$-probability 1, for all but a finite number of indices N, we get

$$c\,\frac{\log N}{N}\leqslant\frac{1}{N}\log\binom{N^+}{N^+-2k}\leqslant c\varepsilon\log\frac{1}{\varepsilon} \tag{4.13}$$

It remains to consider the square root term in (4.4), when $|1-(|m^+|)/(\rho_N^+)| \geqslant \varepsilon \geqslant 1/N$ we have, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\left(1-\frac{(m^+)^2}{\rho_N^+}\right)^{-1/2} \leqslant e^{cN[\varepsilon \log 1/\varepsilon]} \tag{4.14}$$

Collecting (4.10), (4.11), (4.13) and (4.14) and making easy estimates, we get the Lemma 3.3.

Proof of the Lemma 3.2. We have to estimate $\mathscr{G}_N(\mathbb{1}_{T_1^{o,N}})$ from below. We have

$$\mathscr{G}_N(\mathbb{1}_{T_1^{o,N}}) \geqslant \frac{e^{-\beta N[\inf_{m_n \in T_1^{o,N}} \mathscr{F}(m_N)]}}{(2N+1)^2 e^{-\beta N[\inf_{i=1,2} \inf_{m_N \in \bar{T}_i^N} \mathscr{F}_N(m_N)]}} \tag{4.15}$$

Using the Lemma 3.3, we get, for all $\delta > 0$ with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\inf_{i=1,2} \inf_{m_N \in \bar{T}_i^N} \mathscr{F}_N(m_N) \geqslant \mathscr{F}(m_1) - \delta \tag{4.16}$$

since the two minima m_1 and m_2 of $\mathscr{F}$ are quadratic, and $\mathscr{F}(m_1) = \mathscr{F}(m_2)$ moreover

$$\inf_{m_n \in T_1^{o,N}} \mathscr{F}(m_N) \leqslant \mathscr{F}(m_1) + \delta \tag{4.17}$$

therefore inserting (4.16) and (4.17) in (4.15) and making simplifications, we get, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\mathscr{G}_N(\mathbb{1}_{T_1^{o,N}}) \geqslant (2N+1)^{-2} e^{-2\beta N\delta} \tag{4.18}$$

This ends the proof of the Lemma 3.2.

4.1. Convergence of the Measure $\mathscr{G}_N$

In this subsection we consider the static problem related to the convergence of $\mathscr{G}_N$. Similar estimates where done in refs. 3, 4, and 37 but without such an almost sure control and for the empirical magnetization instead of the two empirical parameters $m_N^1(\sigma)$ and $m_N^2(\sigma)$.

We consider the Laplace transform $\mathscr{G}_N[\exp(\zeta_1 m_N^+(\sigma) + \zeta_2 m_N^-(\sigma)]$ and we want to study the convergence to $\alpha e^{\zeta \cdot m_1} + (1-\alpha) e^{\zeta \cdot m_2}$ where α are random variables.

It is convenient to study the following unnormalized version of the previous Laplace transform.

$$L_N(\zeta) = \frac{1}{2^N} \sum_{\sigma \in \mathscr{S}_N} e^{\zeta_1 m_N^+(\sigma) + \zeta_2 m_N^-(\sigma)} e^{\beta N (m_N^+(\sigma) + m_N^-(\sigma))^2 + \beta\theta \sum_{i=1}^N h_i \sigma_i} \tag{4.19}$$

The main result of this section is the following lemma

Lemma 4.1. For all $\varepsilon > 0$, on a set Ω_N, of $\mathbb{P}$-probability not smaller than $1 - 2e^{-\varepsilon^2/2 (\log N)^2}$,

$$\frac{L_N(\zeta)}{L_N(0)} = [e^{(\zeta \cdot m_1)} \alpha_N(\beta, \theta) + (1 - \alpha_N(\beta, \theta))\, e^{(\zeta \cdot m_2)}] \\ \times \left[\left(1 \pm c\varepsilon^3 \frac{(\log N)^3}{\sqrt{N}} \right) \right] e^{\pm \|\zeta\|_2^2/2N} \tag{4.20}$$

where

$$\alpha_N \equiv \alpha_N(\beta, \theta) = \frac{e^{(S_N/2)\, g(z^*)}}{e^{(S_N/2)\, g(z^*)} + e^{-(S_N/2)\, g(z^*)}} \tag{4.21}$$

and $S_N \equiv \sum_{i=1}^N h_i$. Before proving this lemma, let us mention its important consequences. It implies that the difference $\mathscr{G}_N - (\alpha_N \delta_{m_1} + (1 - \alpha_N)\, \delta_{m_2})$ converges weakly to zero, $\mathbb{P}$-almost surely. The argument is the following: Let $B = \bigcup_N \bigcap_{M \geqslant N} \Omega_M$, it follows from the first Borel-Cantelli lemma that $\mathbb{P}(B) = 1$. Let us denote $\nu_N = \alpha_N \delta_{m_1} + (1 - \alpha_N)\, \delta_{m_2}$. Since the set of measure on $[-1, +1]^2$, non necessarily positive but with total mass bounded by 2, is weakly relatively compact, the sequence $\mathscr{G}_N - \nu_N$ has convergent subsequences. That is, for all $\omega \in B$, for all $\nu_\infty \in \mathscr{C}\mathscr{L}(\mathscr{G}_N - \nu_N)$, the cluster set of the sequence $\mathscr{G}_N - \nu_N$, for all $\varepsilon > 0$, for all ϕ that are continuous real valued function on $[-1, +1]^2$, we can find $N_k = N_k(\omega, \varepsilon, \phi)$ such that

$$|\mathscr{G}_{N_k}(\phi) - \nu_{N_k}(\phi) - \nu_\infty(\phi)| \leqslant \varepsilon \tag{4.22}$$

This is true in particular, for $\phi = e^{\zeta \cdot m}$, therefore, using (4.20), we get that for all $\nu_\infty \in \mathscr{C}\mathscr{L}(\mathscr{G}_N - \nu_N)$ the Laplace transform $\nu_\infty(e^{\zeta \cdot m}) = 0$ on B. This is also true, for a given real sequence ζ_p that converges to zero and that is in, say $0 \leqslant \operatorname{Re} \zeta \leqslant 1$. On the other hand using (4.20), it is immediate that for all cluster points ν_∞, $\nu_\infty(e^{\zeta m})$ is in fact an analytic function in the variable ζ in the strip $0 \leqslant \operatorname{Re} \zeta \leqslant 1$. Since $\nu_\infty(e^{\zeta_p m}) = 0$ for a real sequence ζ_p that converge to zero, $\nu_\infty(e^{\zeta m})$ vanish on the strip, in particular, on the pure imaginary axis, therefore all the Fourier transforms of all cluster point are identically zero, and this implies that all the cluster points are equal to the null measure, $\mathbb{P}$-almost surely.

Proof of the Lemma 4.1. Using the formula of the Laplace transform of a Gaussian measure and making easy computations we get

$$L_N(\zeta) = \sqrt{\frac{\beta N}{2\pi}} \int e^{-\beta N \Phi_N(z)}\, dz \tag{4.23}$$

where

$$\Phi_N(z) = \frac{z^2}{2} - \frac{\rho_N^+}{\beta} \log\cosh\left[\beta z + \beta\theta + \frac{\zeta_1}{N}\right] - \frac{\rho_N^-}{\beta} \log\cosh\left[\beta z - \beta\theta + \frac{\zeta_2}{N}\right] \tag{4.24}$$

Let us denote by

$$\mathscr{F}^*(z) \equiv \frac{z^2}{2} - \frac{1}{2\beta} \log\cosh[\beta z + \beta\theta] - \frac{1}{2\beta} \log\cosh[\beta z - \beta\theta] \tag{4.25}$$

and

$$\Delta\mathscr{F}_1^*(z) \equiv -\left(\frac{\rho_N^+}{\beta} - \frac{1}{2\beta}\right) \log\cosh[\beta z + \beta\theta] - \left(\frac{\rho_N^-}{\beta} - \frac{1}{2\beta}\right) \log\cosh[\beta z - \beta\theta] \tag{4.26}$$

Note that if we call $S_N = \sum_{i=1}^N h_i$ we get

$$\begin{aligned} \Delta\mathscr{F}_1^*(z) &= -\frac{S_N}{2N\beta} [\log\cosh(\beta z + \beta\theta) - \log\cosh(\beta z - \beta\theta)] \\ &\equiv -\frac{S_N}{2N\beta} g(z) \end{aligned} \tag{4.27}$$

The fundamental fact is that $g(-z) = -g(z)$ for all $z \in \mathbb{R}$. Let us introduce the following quantities

$$\Delta\mathscr{F}_2^*(z, \zeta) = \Delta\mathscr{F}_3^*(z, \zeta) + \Delta\mathscr{F}_4^*(z, \zeta) \tag{4.28}$$

where

$$\Delta\mathscr{F}_3^*(z,\zeta) \equiv -\left(\frac{\rho_N^+}{\beta}-\frac{1}{2\beta}\right)\left[\log\cosh\left(\beta z+\beta\theta+\frac{\zeta_1}{N}\right)-\log\cosh(\beta z+\beta\theta)\right]$$
$$-\left(\frac{\rho_N^-}{\beta}-\frac{1}{2\beta}\right)\left[\log\cosh\left(\beta z-\beta\theta+\frac{\zeta_2}{N}\right)-\log\cosh(\beta z-\beta\theta)\right] \tag{4.29}$$

and

$$\Delta\mathscr{F}_4^*(z,\zeta) \equiv -\frac{1}{2\beta}\left[\log\cosh\left(\beta z+\beta\theta+\frac{\zeta_1}{N}\right)-\log\cosh(\beta z+\beta\theta)\right]$$
$$-\frac{1}{2\beta}\left[\log\cosh\left(\beta z-\beta\theta+\frac{\zeta_2}{N}\right)-\log\cosh(\beta z-\beta\theta)\right] \tag{4.30}$$

Let us first estimate $\Delta\mathscr{F}_4^*(z,\zeta)$. Using the Taylor formula, we have

$$\left|\Delta\mathscr{F}_4^*(z,\zeta)+\frac{\zeta_1}{2N\beta}\tanh(\beta z+\beta\theta)+\frac{\zeta_2}{2N\beta}\tanh(\beta z-\beta\theta)\right|$$
$$\leqslant\frac{(\zeta_1^2+\zeta_2^2)}{\beta N^2}\sup_{x\in\mathbb{R}}\frac{1}{\cosh^2(x)}\leqslant\frac{(\zeta_1^2+\zeta_2^2)}{\beta N^2} \tag{4.31}$$

Therefore, if we denote

$$\mathscr{T}(z,\zeta) \equiv -\frac{\zeta_1}{2}\tanh(\beta z+\beta\theta)-\frac{\zeta_2}{2}\tanh(\beta z-\beta\theta) \tag{4.32}$$

we get

$$-\frac{\|\zeta\|_2^2}{2N}+\mathscr{T}(z,\zeta)\leqslant N\beta[\Delta\mathscr{F}_4^*(z,\zeta)]\leqslant+\frac{\|\zeta\|_2^2}{2N}+\mathscr{T}(z,\zeta) \tag{4.33}$$

By similar arguments we have, for some positive constant c

$$|\Delta\mathscr{F}_3^*(z,\zeta)|\leqslant c\beta\frac{|S_N|}{N} \tag{4.34}$$

The probabilistic estimate we use is the following, for all $\varepsilon > 0$

$$\mathbb{P}\left[\frac{|S_N|}{N} \geqslant \varepsilon \frac{(\log N)}{\sqrt{N}}\right] \leqslant 2e^{-\varepsilon^2/2\,(\log N)^2} \tag{4.35}$$

Therefore, for all $\varepsilon > 0$, with a $\mathbb{P}$-probability not smaller than $1 - 2e^{-\varepsilon^2/2\,(\log N)^2}$, we have, uniformly with respect to $z \in \mathbb{R}$

$$e^{-\|\zeta\|_2^2/2N} e^{-\varepsilon c(\log N/\sqrt{N})} \leqslant e^{\beta N[|\Delta \mathscr{F}_2^*(z,\zeta)|]} e^{-\mathscr{T}(z,\zeta)} \leqslant e^{\|\zeta\|_2^2/2N} e^{\varepsilon c(\log N/\sqrt{N})} \tag{4.36}$$

That is, we can replace in (4.23) $\Phi_N(z)$ by $\mathscr{F}^*(z) + \Delta \mathscr{F}_1^*(z) + \mathscr{T}(z,\zeta)/N\beta$ in the sense that if we call

$$L_N^1(\zeta) \equiv \sqrt{\frac{\beta N}{2\pi}} \int e^{-\beta N[\mathscr{F}^*(z) + \Delta \mathscr{F}_1^*(z) + \mathscr{T}(z,\zeta)/N\beta]}\, dz \tag{4.37}$$

then, for all $\varepsilon > 0$, with a $\mathbb{P}$-probability not smaller than $1 - 2e^{-\varepsilon^2/2\,(\log N)^2}$, we have

$$e^{-\|\zeta\|_2^2/2N} e^{-\varepsilon c(\log N/\sqrt{N})} \leqslant \frac{L_N(\zeta)}{L_N^1(\zeta)} \leqslant e^{\|\zeta\|_2^2/2N} e^{\varepsilon c(\log N/\sqrt{N})} \tag{4.38}$$

It is easy to check that in the region of parameters β, θ we consider, the function $\mathscr{F}^*(z)$ has three critical points, 0, z^*, $-z^*$, where z^* is the solution of the equation

$$z = \tfrac{1}{2}\tanh(\beta z + \beta\theta) + \tfrac{1}{2}\tanh(\beta z - \beta\theta) \tag{4.39}$$

The important fact is that

$$\frac{\partial^2 \mathscr{F}^*(z)}{\partial z^2} = 1 - \frac{\beta}{2\cosh^2(\beta z + \beta\theta)} - \frac{\beta}{2\cosh^2(\beta z - \beta\theta)} \tag{4.40}$$

satisfies $(\partial^2 \mathscr{F}^*(z)/\partial z^2) = (\partial^2 \mathscr{F}^*(-z)/\partial z^2)$ for all $z \in \mathbb{R}$. Moreover, we recall that $m_1^+ = \frac{1}{2}\tanh(\beta z^* + \beta\theta)$, $m_1^- = \frac{1}{2}\tanh(\beta z^* - \beta\theta)$ and (4.32), therefore we have

$$\begin{aligned} \mathscr{T}(z^*,\zeta) &= \zeta_1 m_1^+ + \zeta_2 m_1^- \\ \mathscr{T}(-z^*,\zeta) &= \zeta_1 m_2^+ + \zeta_2 m_2^- \end{aligned} \tag{4.41}$$

It is useful to write

$$L_N^1(\zeta)=e^{-\beta N\mathscr{F}^*(z^*)}\sqrt{\frac{\beta N}{2\pi}}$$
$$\times\left[e^{\zeta_1 m_1^+ +\zeta_2 m_1^-}e^{(S_N/2)\,g(z^*)}I_+ + e^{\zeta_1 m_2^+ +\zeta_2 m_2^-}e^{(-S_N/2)\,g(z^*)}I_-\right] \tag{4.42}$$

Where I_+ is just

$$I_+\equiv\sqrt{\frac{\beta N}{2\pi}}\int_{z\geqslant 0}dz\,e^{-N\beta[\mathscr{F}^*(z)-\mathscr{F}^*(z^*)]}e^{S_N/2\,[g(z)-g(z^*)]}e^{[\mathscr{T}(z,\zeta)-\mathscr{T}(z^*,\zeta)]} \tag{4.43}$$

and I_- is the very same integral but over $z\leqslant 0$.

We need at this point a careful study of this integral. First we want to control an almost sure behavior, moreover, what we are interested in is the ratio $L_N^1(\zeta)/L_N^1(0)$ and we want to take into account of some nice cancellations that occurs. This is not really difficult but it is rather long to make all the details. Let us first notice that, by using the Taylor theorem, we get

$$\mathscr{F}^*(z)-\mathscr{F}^*(z^*)-\frac{S_N}{2\beta N}(g(z)-g(z^*))$$
$$=\frac{1}{2}(z-z^*)^2\,C_N-(z-z^*)\,\Delta_{m_1}\frac{S_N}{N}\pm\frac{4}{3}(z-z^*)^3\,\beta^2\left(1+\frac{S_N}{N}\right) \tag{4.44}$$

where

$$C_N\equiv\left[\mathscr{F}^{*\prime\prime}(z^*)+\frac{\beta S_N}{2N}\left(\frac{1}{\cosh^2(\beta z^*+\beta\theta)}-\frac{1}{\cosh^2(\beta z^*-\beta\theta)}\right)\right] \tag{4.45}$$

and

$$\Delta_{m_1}\equiv(m_1^+-m_1^-) \tag{4.46}$$

Note at this point the important fact that, the quadratic form that appears in (4.44) can be written

$$\frac{1}{2}(z-z^*)^2\,C_N-(z-z^*)\,\Delta_{m_1}\frac{S_N}{N}$$
$$=-\frac{1}{2}\frac{(\Delta_{m_1})^2}{C_N}\left(\frac{S_N}{N}\right)^2+\frac{1}{2}\left((z-z^*)\sqrt{C_N}-\frac{S_N}{N}\frac{\Delta_{m_1}}{\sqrt{C_m}}\right)^2 \tag{4.47}$$

and therefore that the center of the Gaussian is shifted by a random term that is going to zero almost surely, and also the covariance is shifted by such a similar term. Consider, for a given ε_n, to be chosen later

$$I_+(\varepsilon_N) \equiv \sqrt{\frac{\beta N}{2\pi}} \int_{\substack{|z-z^*| \leqslant \varepsilon_n \\ z \geqslant 0}} dz\, e^{-N\beta[\mathscr{F}^*(z) - \mathscr{F}^*(z^*)]} \times e^{S_N/2\,[g(z) - g(z^*)]} e^{[\mathscr{T}(z,\zeta) - \mathscr{T}(z^*,\zeta)]} \tag{4.48}$$

Using (4.44), the term corresponding to $(z-z^*)^3$ gives a contribution which is not bigger than $N\varepsilon_N^3$ this suggest to take $\varepsilon_N = (\tilde{\varepsilon}_N)/(N^{1/3})$ in which case we get that the term $[\mathscr{T}(z,\zeta) - \mathscr{T}(z^*,\zeta)]$ is not bigger than $\beta\ \|\zeta\|_1 \varepsilon_N$. Therefore we get, for all $\varepsilon > 0$, with a $\mathbb{P}$-probability not smaller than $1 - 2e^{-\varepsilon^2/2\,(\log N)^2}$,

$$I_+(\varepsilon_N) = e^{N\beta/2\,(\Delta_{m_1})^2/C_N\,(S_N/N)^2} e^{\pm 4\beta^3 \tilde{\varepsilon}_N/3\,(1 + \varepsilon(\log N/\sqrt{N}))} \times \sqrt{\frac{\beta N}{2\pi}} \int_{\substack{|z-z^*| \leqslant \varepsilon_n \\ z \geqslant 0}} dz\, e^{-N\beta/2\,((z-z^*)\sqrt{C_N} - (S_N/N)(\Delta_{m_1}/\sqrt{C_m}))^2} \tag{4.49}$$

After a simple change of variables, it remains to estimate a Gaussian integral

$$\gamma_1 \equiv \frac{1}{\sqrt{C_N}} \int e^{-x^2/2} \mathbb{1}_{B_N}(x)\, dx \tag{4.50}$$

here $B_N = B_N^1 \cap B_N^2$ with

$$B_N^1 \equiv \left\{ -N^{1/6} \left(\tilde{\varepsilon}_N \sqrt{\beta C_N} + \frac{S_N \Delta_{m_1}}{N^{2/3}\sqrt{\beta C_N}} \right) \leqslant x \leqslant N^{1/6} \left(\tilde{\varepsilon}_N \sqrt{\beta C_N} - \frac{S_N \Delta_{m_1}}{N^{2/3}\sqrt{\beta C_N}} \right) \right\} \tag{4.51}$$

and

$$B_N^2 \equiv \left\{ x \geqslant -\sqrt{N} \left(z^* \sqrt{\beta C_N} + \frac{S_N \Delta_{m_1}}{N\sqrt{\beta C_N}} \right) \right\} \tag{4.52}$$

On the set $G_N \equiv \{|S_N/N| \leqslant \varepsilon(\log N/\sqrt{N})\}$, it is easy to check that

$$z^* \sqrt{\beta C_N} + \frac{S_N \Delta_{m_1}}{N\sqrt{\beta C_N}} = z^* \sqrt{\beta \mathscr{F}^{*\prime\prime}(z^*)}\left(1 \pm c_1(\beta, \theta)\, \varepsilon \frac{\log N}{\sqrt{N}}\right) \tag{4.53}$$

and therefore, by inspection on the set G_N, we have

$$\gamma_1 = \frac{1}{\sqrt{\beta \mathscr{F}^{*\prime\prime}(z^*)}}\left(1 \pm c\varepsilon \frac{\log N}{\sqrt{N}}\right)(1 \pm e^{-cN^{1/3}}) \tag{4.54}$$

Making similar computations, it is easy to check that on the set G_N

$$e^{N\beta/2\,(\Delta_{m_1})^2/C_N\,(S_N/N)^2} = e^{N\beta/2\,(\Delta_{m_1})^2/\mathscr{F}^{*\prime\prime}(z^*)\,(S_N/N)^2} e^{\pm c\varepsilon^3\,((\log N)^3/\sqrt{N})} \tag{4.55}$$

for some positive constant c.

Collecting (4.54) and (4.55) we get that on G_N

$$I_+(\varepsilon_N) = \frac{e^{N\beta/2\,(\Delta_{m_1})^2/\mathscr{F}^{*\prime\prime}(z^*)\,(S_N/N)^2}}{\sqrt{\beta \mathscr{F}^{*\prime\prime}(z^*)}}(1 \pm \tilde{\varepsilon}_N^3)\left(1 \pm c\varepsilon^3 \frac{(\log N)^3}{\sqrt{N}}\right) \tag{4.56}$$

Exactly the same computations can be done for $I_-(\varepsilon_N)$ and we get on G_N

$$I_-(\varepsilon_N) = \frac{e^{N\beta/2\,(\Delta_{m_2})^2/\mathscr{F}^{*\prime\prime}(-z^*)\,(S_N/N)^2}}{\sqrt{\beta \mathscr{F}^{*\prime\prime}(-z^*)}}(1 \pm \tilde{\varepsilon}_N^3)\left(1 \pm c\varepsilon^3 \frac{(\log N)^3}{\sqrt{N}}\right) \tag{4.57}$$

The point is now that $\Delta_{m_1} = -\Delta_{m_2}$ and $\mathscr{F}^{*\prime\prime}(z^*) = \mathscr{F}^{*\prime\prime}(-z^*)$ as it can be checked easily.

It remains to estimate two Gaussian integrals,

$$\begin{aligned} I_\pm^>(\varepsilon_N) \equiv \sqrt{\frac{\beta N}{2\pi}} \int_{\substack{|z - \pm z^*| \geqslant \varepsilon_N \\ z \geqslant 0}} dz\, e^{-N\beta[\mathscr{F}^*(z) - \mathscr{F}^*(z^*)]} \\ \times e^{S_N/2\,[g(z) - g(z^*)]} e^{[\mathscr{T}(z,\zeta) - \mathscr{T}(z^*,\zeta)]} \end{aligned} \tag{4.58}$$

We consider $I_+^>(\varepsilon_N)$ the other one being similar. Since $\mathscr{F}^*$ has a quadratic minimum at ζ^*, there exists a constant $c(\beta, \theta)$ such that

$$[\mathscr{F}^*(z) - \mathscr{F}^*(z^*)] \geqslant c(\beta, \theta)(z - z^*)^2\, c \tag{4.59}$$

and a constant $\tilde{c}(\beta, \theta)$ such that

$$|g(z) - g(z^*)| \leqslant \tilde{c}(\beta, \theta)\, |z - z^*| \tag{4.60}$$

therefore we get

$$I_{\pm}^{>}(\varepsilon_N) \leqslant \sqrt{\frac{\beta N}{2\pi}} \int_{|z - \pm z^*| \geqslant \varepsilon_N} e^{-N\beta c(\beta, \theta)\, |z - z^*|\, [(\tilde{\varepsilon}_N/N^{1/3}) - (S_N \tilde{c}(\beta, \theta)/N\beta c(\beta, \theta)]} \tag{4.61}$$

On the set G_N, we have

$$I_{\pm}^{>}(\varepsilon_N) \leqslant \sqrt{\frac{\beta N}{2\pi}} \int_{|z - \pm z^*| \geqslant \varepsilon_N} e^{-N^{2/3}\beta c(\beta, \theta)\, |z - z^*|\, [\tilde{\varepsilon}_N - (\varepsilon \log N \tilde{c}(\beta, \theta)/N^{1/6}\beta c(\beta, \theta))]} \tag{4.62}$$

Therefore if $\tilde{\varepsilon}_N$ is chosen in such a way that

$$\tilde{\varepsilon}_N = (\ell + 1)\, \frac{\varepsilon \log N \tilde{c}(\beta, \theta)}{N^{1/6} \beta c(\beta, \theta)} \tag{4.63}$$

for some ℓ to be chosen later, we get after some easy estimates that

$$I_{\pm}^{>}(\varepsilon_N) \leqslant c(\ell \log N)^{-1}\, e^{-\ell^2 (\log N)^2\, \tilde{c}(\beta, \theta)/c(\beta, \theta)} \tag{4.64}$$

Therefore taking ℓ large enough, on G_N, we have

$$\frac{L_N^1(\zeta)}{L_N^1(0)} = [e^{(\zeta \cdot m_1)} \alpha_N(\beta, \theta) + (1 - \alpha_N(\beta, \theta))\, e^{(\zeta \cdot m_2)}] \left[\left(1 \pm c\varepsilon^3 \frac{(\log N)^3}{\sqrt{N}} \right) \right] \tag{4.65}$$

here

$$\alpha_N(\beta, \theta) = \frac{e^{S_N/2\, g(z^*)}}{e^{S_N/2\, g(z^*)} + e^{-S_N/2\, g(z^*)}} \tag{4.66}$$

Recalling (4.38), we get, on G_N

$$\frac{L_N(\zeta)}{L_N(0)} = [e^{(\zeta \cdot m_1)} \alpha_N(\beta, \theta) + (1 - \alpha_N(\beta, \theta))\, e^{(\zeta \cdot m_2)}] \times \left[\left(1 \pm c\varepsilon^3 \frac{(\log N)^3}{\sqrt{N}} \right) \right] e^{\pm \|\zeta\|_2^2/2N} \tag{4.67}$$

which is what we wanted to prove.

5. PROOFS OF THE THEOREMS

Proof of the Theorem 2.1. Let $\alpha > \Delta\mathscr{F}$, then using (2.47), we get

$$P_{m_N}[\tau_N > e^{\alpha\beta N}] = \sum_{i \geq 1} \varphi_i^{N,K}(m_N)\, e^{-\Lambda_i^{N,K} e^{\alpha\beta N}} \mathscr{G}_N^1(\varphi_i^{N,K}) \tag{5.1}$$

using now (3.86), (3.59) together with $\Lambda_k^{N,K} \geq \Lambda_2^{N,K}$ for all $k \geq 2$ and (3.50) we get, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$P_{m_N}[\tau_N > e^{\alpha\beta N}] \leq c e^{\beta N[\operatorname{Osc} \mathscr{F} + \delta]} e^{-\exp[\beta N(\alpha - \Delta\mathscr{F} - \delta)]} \tag{5.2}$$

which goes super-exponentially fast to zero if δ is chosen small enough. Now, let $\alpha < \Delta\mathscr{F}$, we want to prove that

$$\lim_{N \nearrow \infty} P_{m_N}[\tau_N > e^{\alpha\beta N}] = 1 \tag{5.3}$$

Note first that

$$\begin{aligned} &|P_{m_N}[\tau_N > e^{\alpha\beta N}] - 1 + 1 - \varphi_1^{N,K}(m_N)\, e^{-\Lambda_1^{N,K} \exp[\alpha\beta N]} \mathscr{G}_1(\varphi_1^{N,K})| \\ &\quad \leq \sup_{m_N} |(e^{e^{\alpha\beta N} \mathscr{L}_N^K}(1 - \varphi_1^{N,K}))(m_N)| \end{aligned} \tag{5.4}$$

Using now the L^∞, L^2 estimates (3.86), we get that the right hand side of (5.4) does not exceed

$$e^{\beta N \operatorname{Osc} \mathscr{F}} \|e^{e^{\alpha\beta N} \mathscr{L}^K}(1 - \varphi_1^{N,K})\|_2 \tag{5.5}$$

where $\|\cdot\|_2$ is the $L^2(T_1^N, \mathscr{G}_1^N)$ norm. Using now the spectral decomposition of the semigroup we get

$$\|e^{e^{\alpha\beta N} \mathscr{L}^K}(1 - \varphi_1^{N,K})\|_2 \leq e^{-\Lambda_2^{N,K} e^{\alpha\beta N}} \tag{5.6}$$

Using now, (3.59) we get

$$\Lambda_2^{N,K} e^{\alpha\beta N} \geq e^{(\alpha - \delta)\beta N} \tag{5.7}$$

that diverges with N. Therefore the right hand side of (5.4) does not exceed

$$e^{\beta N \operatorname{Osc} \mathscr{F}} e^{-e^{(\alpha - \delta)\beta N}} \tag{5.8}$$

that goes to 0 when $N \uparrow \infty$.

On the other hand, using the Proposition 3.6 and $\lim_{N\nearrow\infty} e^{-\Lambda_1^{N,K} e^{\alpha\beta N}} = 1$, we get (5.3) and this end the proof of the Theorem 2.1.

Proof of the Theorem 2.2. Using the spectral decomposition we get

$$P_{m_N}[\tau_N > t(\Lambda_1^{N,K})^{-1}] = \varphi_1^{N,K}(m_N)\, e^{-t}\mathscr{G}_N^1(\varphi_1^{N,K}) + \sum_{i\geq 2} \varphi_i^{N,K}(m_N)\, e^{-t(\Lambda_i^{N,K}/\Lambda_1^{N,K})}\mathscr{G}_N^1(\varphi_i^{N,K}) \tag{5.9}$$

Using the Proposition 3.6, (3.73) and the Proposition 3.7 together with the following estimates, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N

$$\sup_{i\geq 2} e^{-t(\Lambda_i^{N,K}/\Lambda_1^{N,K})} \leq e^{[-\exp(\beta N(\Delta\mathscr{F} - 2\delta))]} \tag{5.10}$$

which follows from the Proposition 3.5 and (3.59), we get (2.49). We have also used the fact that $\Lambda_i^{N,K} \geq \Lambda_2^{N,K}$ for all $i \geq 2$.

Proof of the Theorem 2.3. The Theorem 2.3 is nothing but the Proposition 3.5

Proof of the Theorem 2.4. Recall that the starting point of the process $m_N(t)$ is in T_1^N. Let Ψ be a continuous real valued bounded function on $\mathscr{M}_N$, then if $t_N = e^{\alpha\beta N}$ with $\alpha < \Delta\mathscr{F}$,

$$E_{m_N}[\Psi(m_N(t_N))] = E_{m_N}[\Psi(m_N(t_N)), t_N < \tau_N] + E_{m_N}[\Psi(m_N(t_N)), t_N \geq \tau_N] \tag{5.11}$$

The last term is bounded from above by

$$\sup_{m_N \in \mathscr{M}_N} |\Psi(m_N)|\, P_{m_N}[t \geq \tau_N] \tag{5.12}$$

and the last term goes to zero by (5.2). It remains to consider the first term in the right hand side of (5.11). Since on $\{t_N < \tau_N\}$, the process $m_N(t)$ and the process $m_N^R(t)$ reflected on $\bar{T}_1^N$ are equal we have, using (5.12),

$$\begin{aligned} E_{m_N}[\Psi(m_N(t_N)), t_N < \tau_N] &= E_{m_N}[\Psi(m_N^R(t_N)), t_N < \tau_N] \\ &= E_{m_N}[\Psi(m_N^R(t_N))] \pm \sup_{m_N \in \mathscr{M}_N} |\Psi(m_N)|\, P_{m_N}[t \geq \tau_N] \end{aligned} \tag{5.13}$$

Now using the spectral decomposition of $\mathscr{L}_N^R$, we get

$$|E_{m_N}[\Psi(m_N^R(t_N))] - e^{-tA_0^{N,R}}\varphi_0^{N,R}(m_N)\,\mathscr{G}_N^1(\Psi)| \leqslant \sup_{m_N \in \mathscr{M}_N} |\Psi(m_N)|\, e^{\beta N[\mathrm{Osc}(\mathscr{F})-\delta]} \sum_{i \geqslant 1} e^{-t_N A_i^{N,R}} \tag{5.14}$$

where we have used (3.86). Now since, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$A_i^{N,R} \geqslant e^{-\delta\beta N} \tag{5.15}$$

for all $i \geqslant 1$, we get

$$e^{\beta N[\mathrm{Osc}(\mathscr{F})-\delta]} \sum_{i \geqslant 1} e^{-t_N A_i^{N,R}} \leqslant (2N+1)^2\, e^{-e^{[\alpha\beta-\delta]N}} e^{\beta N[\mathrm{Osc}(\mathscr{F})-\delta]} \tag{5.16}$$

which goes to zero when $N \nearrow \infty$. On the other hand $\varphi_0^{N,R} = 1$ and $A_0^{N,R} = 0$ therefore we get immediatly, $\mathbb{P}$-almost surely

$$\lim_{N \nearrow \infty} |E_{m_N}[\Psi(m_N(t_N)),\, t_N \leqslant \tau_N] - \mathscr{G}_N^1(\Psi)| = 0 \tag{5.17}$$

Therefore to get (2.51) it is enough to prove that, with $\mathbb{P}$-probability 1,

$$\lim_{N \nearrow \infty} |\mathscr{G}_N^1(\Psi) - \Psi(m_1)| = 0 \tag{5.18}$$

but this is easy to prove.

Proof of the Theorem 2.5. Let $t_N = e^{\alpha\beta N}$ with $\alpha > \Delta\mathscr{F}$. We use the spectral decomposition of the semigroup associated to the process $m_N(t)$, to get

$$E_{m_N}[\Psi(m_N(t))] = \mathscr{G}_N(\Psi) + \sum_{i \geqslant 1} e^{-t_N A_i^N} \varphi_i(m_N)\, \mathscr{G}_N(\varphi_i \Psi) \tag{5.19}$$

Now, since by Schwarz inequality,

$$\mathscr{G}_N(\varphi_i \Psi) \leqslant \|\Psi\|_\infty \, \|\varphi_i\|_2 = \|\Psi\|_\infty \tag{5.20}$$

we get, for all $\delta > 0$, with $\mathbb{P}$-probability 1, for all but a finite number of indices N,

$$\sup_{m_N \in T_1^N} |E_{m_N}[\Psi(m_N(t))] - \mathscr{G}_N(\Psi)| \leqslant (2N+1)^2\, e^{-e^{[\beta N(\alpha - \Delta\mathscr{F} - \delta)]}} e^{\beta N\,\mathrm{Osc}(\mathscr{F})}\, \|\Psi\|_\infty \tag{5.21}$$

and this end the proof of the Theorem 2.5.

Proof of the Theorem 2.6. Let us start with the formula (2.46), we have, for all $t' > 0$

$$E_{m_N}[\Psi(m_N(t'))] = \mathscr{G}_N(\Psi) + \varphi_1(m_N)\, e^{-\Lambda_1^N t'} \mathscr{G}_N(\Psi \varphi_1) + \sum_{i \geqslant 2} \varphi_i(m_N)\, e^{-\Lambda_i^N t'} \mathscr{G}_N(\Psi \varphi_i) \tag{5.22}$$

In particular, if we take $t' = t/\Lambda_1^N$ with $t > 0$, recalling (3.70), we get

$$\left| E_{m_N}\left[\Psi\left(m_N\left(\frac{t}{\Lambda_1^N} \right) \right) \right] - (\mathscr{G}_N(\Psi) + \varphi_1(m_N)\, e^{-t} \mathscr{G}_N(\Psi \varphi_1)) \right| \leqslant \left| \sum_{i \geqslant 2} \varphi_i(m_N)\, e^{-(\Lambda_i^N/\Lambda_1^N)t} \mathscr{G}_N(\Psi \varphi_i) \right|$$

Note that the terms $e^{-(\Lambda_i^N/\Lambda_1^N)t}$ are super-exponentially small in N, therefore using the Proposition 3.7, the last sum in the right hand side of (5.23) is also super-exponentially small in N.

To estimate the term $\varphi_1(m_N)\, \mathscr{G}_N(\Psi \varphi_1)$ in (5.23), we make a short time argument as follows:

If in (5.22), we take $t' = e^{\alpha \beta N}$ with $\alpha < \Delta \mathscr{F}$, we get

$$\left| E_{m_N}\left[\Psi\left(m_N\left(\frac{t}{\Lambda_1^N} \right) \right) \right] - (\mathscr{G}_N(\Psi) + \varphi_1(m_N)\, \mathscr{G}_N(\Psi \varphi_1)) \right| \leqslant e^{-c e^{\alpha \beta N}} \tag{5.24}$$

for some positive constant c. Using the Theorem 2.4, the formula (5.23), and the triangle inequality, we get for all $\varepsilon > 0$, with $\mathbb{P}$-Probability 1, for all but a finite number of indices N,

$$|\Psi(m_1) - \mathscr{G}_N(\Psi) - \varphi_1(m_N)\, \mathscr{G}_N(\Psi \varphi_1)| \leqslant 2\varepsilon \tag{5.25}$$

therefore, for all $\varepsilon > 0$, with $\mathbb{P}$-Probability 1, for all but a finite number of indices N,

$$\left| E_{m_N}\left[\Psi\left(m_N\left(\frac{t}{\Lambda_1^N} \right) \right) \right] - (\mathscr{G}_N(\Psi) + e^{-t}(-\Psi(m_1) + \mathscr{G}_N(\Psi))) \right| \leqslant 3\varepsilon \tag{5.26}$$

and this ends the proof of the Theorem 2.6.

Proof of the Theorem 2.7. The Theorem 2.7 is nothing but the Proposition 3.1.

Proof of the Theorem 2.8. The proof of the Theorem 2.8 is immediate from (4.65) and (4.66).

Proof of the Theorem 2.9. The proof of the Theorem 2.9, is also easy. Since for all k $\lim_{N\uparrow\infty} \mathbb{P}[|S_N| \leqslant k] = 0$, we get, for all $\varepsilon > 0$, $\lim_{N\uparrow\infty} \mathbb{P}[\varepsilon \leqslant \alpha_N(\beta, \theta) \leqslant 1-\varepsilon] = 0$. By symmetry we have $\mathbb{P}[\alpha_N(\beta,\theta) > 1-\varepsilon] = \mathbb{P}[\alpha_N(\beta,\theta) < \varepsilon]$, therefore we get $\lim_{N\uparrow\infty} \mathbb{P}[\alpha_N(\beta,\theta) \leqslant \varepsilon] = 1/2 = \lim_{N\uparrow\infty} \mathbb{P}[\alpha_N(\beta,\theta) \geqslant 1-\varepsilon]$, which is what we wanted to prove.

APPENDIX. THE DIRICHLET FORM OF $\mathscr{L}_N$

In this appendix we want to check that the Dirichlet form defined in (2.25) is the one associated to $\mathscr{L}_N$.

To simplify notations let us introduce more definitions. For any ϕ real valued function on $\mathscr{L}_N$, and $(\varepsilon_1, \varepsilon_2) \in \{-1, +1\}^2$, let us call

$$[\nabla_{\varepsilon_1}^{\varepsilon_2}\phi](m_N^+(\sigma), m_N^-(\sigma)) \equiv \left[\phi\left(m_N^+(\sigma) - \frac{(1+\varepsilon_1)\,\varepsilon_2}{N}, m_N^-(\sigma) - \frac{(1-\varepsilon_1)\,\varepsilon_2}{N}\right) - \phi(m_N^+(\sigma), m_N^-(\sigma))\right] \tag{6.1}$$

and note that the upper index $\varepsilon_2 = \pm 1$ corresponds to right or left discrete derivative and $\varepsilon_1 = \pm 1$ correspond to the first or second coordinate.

With these notations the infinitesimal generator $\mathscr{L}_N$ is explicitly given by

$$[\mathscr{L}_N\phi](m_N^+, M_N^-)\, Z_N = \sum_{(\varepsilon_1, \varepsilon_2)} \frac{(\varepsilon_2 m_N^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} [\nabla_{\varepsilon_1}^{\varepsilon_2} f](m_N^+, m_N^-)\, e^{-\beta/2\,[\nabla_{\varepsilon_1}^{\varepsilon_2} H_N(m_N^+, m_N^-)]} \tag{6.2}$$

We will drop out all the N dependences when it is possible, to simplify the formulae. We have

$$\begin{aligned} -\mathscr{G}_N(\Psi[\mathscr{L}_N\phi])\, Z_N &= -\sum_{m^+, m^-} \Psi(m^+, m^-)[\mathscr{L}_N\phi](m^+, m^-)\, e^{-\beta H_N(m^+, m^-)} \\ &\quad\times \binom{N^+}{\left(1+\frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1+\frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}} \\ &= -\sum_{m^+, m^-} \sum_{(\varepsilon_1, \varepsilon_2)} \Psi(m^+, m^-) \binom{N^+}{\left(1+\frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1+\frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}} \\ &\quad\times \frac{(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} [\nabla_{\varepsilon_1}^{\varepsilon_2}\phi](m^+, m^-)\, e^{-\beta/2[\nabla_{\varepsilon_1}^{\varepsilon_2} H_N(m^+, m^-) + 2H_N(m^+, m^-)]} \end{aligned} \tag{6.3}$$

We fix a pair $(\varepsilon_1, \varepsilon_2)$ and perform the change of variables

$$(m^+, m^-) = \left(\tilde{m}^+ + \frac{(1+\varepsilon_1)\,\varepsilon_2}{N}, \tilde{m}^- + \frac{(1-\varepsilon_1)\,\varepsilon_2}{N}\right) \tag{6.4}$$

Note that, this is possible except for $m^{\varepsilon_1} = -\varepsilon_2 \rho_N^{\varepsilon_1}$ (we will be outside $\mathscr{M}_N$) but such a term does not appear because of the nice factor $(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})/2$ in front. Moreover

$$\begin{aligned}
&[\nabla_{\varepsilon_1}^{\varepsilon_2} H_N(m^+, m^-) + 2H_N(m^+, m^-)] \\
&\qquad = [\nabla_{\varepsilon_1}^{-\varepsilon_2} H_N \tilde{m}^+, \tilde{m}^-) + 2H_N(\tilde{m}^+, \tilde{m}^-)]
\end{aligned} \tag{6.5}$$

Therefore we get, for $(\varepsilon_1, \varepsilon_2)$ given

$$\begin{aligned}
&-\sum_{m^+, m^-} \Psi(m^+, m^-) \binom{N^+}{\left(1 + \frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1 + \frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}} \\
&\quad \times \frac{(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} [\nabla_{\varepsilon_1}^{\varepsilon_2} \phi](m^+, m^-) \\
&\quad \times e^{-\beta/2[\nabla_{\varepsilon_1}^{\varepsilon_2} H_N(m^+, m^-) + 2H_N(m^+, m^-)]} \\
&= -\sum_{\tilde{m}^+, \tilde{m}^-} \Psi\left(\tilde{m}^+ + \frac{(1+\varepsilon_1)\,\varepsilon_2}{N}, \tilde{m}^- + \frac{(1-\varepsilon_1)\,\varepsilon_2}{N}\right) \\
&\quad \times \binom{N^+}{\left(1 + \frac{(\tilde{m}^+ + [(1+\varepsilon_1)\,\varepsilon_2)/N]}{\rho_N^+}\right)\frac{N^+}{2}} \\
&\quad \times \binom{N^-}{\left(1 + \frac{\tilde{m}^- + [(1-\varepsilon_1)\,\varepsilon_2]/N}{\rho_N^-}\right)\frac{N^-}{2}} \\
&\quad \times \frac{(\varepsilon_2 \tilde{m}^{\varepsilon_1} + 2/N + \rho_N^{\varepsilon_1})}{2} [-\nabla_{\varepsilon_1}^{-\varepsilon_2} \phi](\tilde{m}^+, \tilde{m}^-) \\
&\quad \times e^{-\beta/2[\nabla_{\varepsilon_1}^{-\varepsilon_2} H_N(\tilde{m}^+, \tilde{m}^-) + 2H_N(\tilde{m}^+, \tilde{m}^-)]}
\end{aligned} \tag{6.6}$$

Now we use the crucial but elementary fact that

$$\frac{(\varepsilon_2 \tilde{m}^{\varepsilon_1} + 2/N + \rho_N^{\varepsilon_1})}{2} \binom{N^+}{\left(1 + \frac{(\tilde{m}^+ + [(1+\varepsilon_1)\varepsilon_2]/N)}{\rho_N^+}\right)\frac{N^+}{2}} \times \binom{N^-}{\left(1 + \frac{\tilde{m}^- + [(1-\varepsilon_1)\varepsilon_2]/N}{\rho_N^-}\right)\frac{N^-}{2}} = \frac{(-\varepsilon_2 \tilde{m}^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} \binom{N^+}{\left(1 + \frac{\tilde{m}^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1 + \frac{\tilde{m}^-}{\rho_N^-}\right)\frac{N^-}{2}} \tag{6.7}$$

therefore the right hand side of (2.39) is equal to

$$- \sum_{\tilde{m}^+, \tilde{m}^-} \Psi\left(\tilde{m}^+ + \frac{(1+\varepsilon_1)\varepsilon_2}{N}, \tilde{m}^- + \frac{(1-\varepsilon_1)\varepsilon_2}{N}\right) \times \binom{N^+}{\left(1 + \frac{\tilde{m}^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1 + \frac{\tilde{m}^-}{\rho_N^-}\right)\frac{N^-}{2}} \frac{(-\varepsilon_2 \tilde{m}^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} \times [-\nabla_{\varepsilon_1}^{-\varepsilon_2} \phi](\tilde{m}^+, \tilde{m}^-)\, e^{-\beta/2[\nabla_{\varepsilon_1}^{-\varepsilon_2} H_N(\tilde{m}^+, \tilde{m}^-) + 2H_N(\tilde{m}^+, \tilde{m}^-)]} \tag{6.8}$$

and we recognize here, that for all but the term Ψ and the minus sign in front of the $\nabla_{\varepsilon_1}^{-\varepsilon_2}$ we have the term corresponding to the right hand side of (2.38) taken for the pair $(\varepsilon_1, -\varepsilon_2)$ therefore keeping half of terms of the right hand side of (2.38) as they are and making the previous change of variables for the other half we get after collecting

$$-\mathscr{G}_N(\Psi[\mathscr{L}_N \phi]) = \frac{1}{2Z_N} \sum_{m^+, m^-} \sum_{(\varepsilon_1, \varepsilon_2)} \frac{(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} [\nabla_{\varepsilon_1}^{\varepsilon_2} \Psi](m^+, m^-)[\nabla_{\varepsilon_1}^{\varepsilon_2} \phi](m^+, m^-) \times \binom{N^+}{\left(1 + \frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1 + \frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}} e^{-\beta/2[\nabla_{\varepsilon_1}^{\varepsilon_2} H_N(m^+, m^-) + 2H_N(m^+, m^-)]} \equiv \mathscr{E}_N(\Psi, \phi) \tag{6.9}$$

Now we want to write the previous formula in a slightly different form to express the last three factors in term of the free energy functional $\mathscr{F}_N$ see (2.17). To do this, we use again the equation (6.7) to get

$$\frac{(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} \binom{N^+}{\left(1 + \frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1 + \frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}}$$

$$= \left(\frac{(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} \binom{N^+}{\left(1 + \frac{m^+}{\rho_N^+}\right)\frac{N^+}{2}} \binom{N^-}{\left(1 + \frac{m^-}{\rho_N^-}\right)\frac{N^-}{2}} \right)^{1/2}$$

$$\times \left(\frac{(-\varepsilon_2 m^{\varepsilon_1} + 2/N + \rho_N^{\varepsilon_1})}{2} \binom{N^+}{\left(1 + \frac{(m^+ - [(1+\varepsilon_1)\varepsilon_2]/N)}{\rho_N^+}\right)\frac{N^+}{2}} \right.$$

$$\left. \times \binom{N^-}{\left(1 + \frac{m^- [(1-\varepsilon_1)\varepsilon_2]/N}{\rho_N^-}\right)\frac{N^-}{2}} \right)^{1/2} \tag{6.10}$$

inserting (6.10) in (6.9) and after identification we get

$$\begin{aligned} &\mathscr{E}_N(\Psi, \phi) \\ &\equiv -\mathscr{G}_N(\Psi[\mathscr{L}_N \phi]) \\ &= \frac{1}{2Z_N} \sum_{m^+, m^-} \sum_{(\varepsilon_1, \varepsilon_2)} [\nabla_{\varepsilon_1}^{\varepsilon_2} \Psi](m^+, m^-)[\nabla_{\varepsilon_1}^{\varepsilon_2} \phi](m^+, m^-) \\ &\quad \times \left(\frac{(\varepsilon_2 m^{\varepsilon_1} + \rho^{\varepsilon_1})}{2} \frac{(-\varepsilon_2 m^{\varepsilon_1} + 2/N + \rho_N^{\varepsilon_1})}{2} \right)^{1/2} \\ &\quad \times e^{-\beta N/2 \, [\nabla_{\varepsilon_1}^{\varepsilon_2} \mathscr{F}_N(m^+, m^-) + 2\mathscr{F}_N(m^+, m^-)]} \end{aligned} \tag{6.11}$$

Note the presence of the terms into the square root. If $m \in \mathscr{M}_N$ and $\tilde{m} \sim m$, recalling (2.32), let us denote

$$\mathscr{N}(\tilde{m}, m) \equiv \frac{(\varepsilon_2 m^{\varepsilon_1} + \rho_N^{\varepsilon_1})}{2} \frac{(-\varepsilon_2 m^{\varepsilon_1} + 2/N + \rho_N^{\varepsilon_1})}{2} \tag{6.12}$$

We write (6.11), in the form

$$\mathscr{E}_N(\Psi, \Psi) = \frac{1}{2Z_N} \sum_{\tilde{m}:\tilde{m}\sim m} (\Psi(\tilde{m}) - \Psi(m))^2 (\tilde{\mathscr{N}}_N(\tilde{m}, m))^{1/2} e^{-\beta N/2 [\mathscr{F}_N(\tilde{m}) + \mathscr{F}_N(m)]} \tag{6.13}$$

which is what we wanted to check.

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